

# Representations of the Hardy Algebra: Absolute Continuity, Intertwiners, and Superharmonic Operators

Paul S. Muhly\*

Department of Mathematics

University of Iowa

Iowa City, IA 52242

e-mail: [muhly@math.uiowa.edu](mailto:muhly@math.uiowa.edu)

Baruch Solel†

Department of Mathematics

Technion

32000 Haifa, Israel

e-mail: [mabaruch@techunix.technion.ac.il](mailto:mabaruch@techunix.technion.ac.il)

June 9, 2010

## Abstract

Suppose  $\mathcal{T}_+(E)$  is the tensor algebra of a  $W^*$ -correspondence  $E$  and  $H^\infty(E)$  is the associated Hardy algebra. We investigate the problem of extending completely contractive representations of  $\mathcal{T}_+(E)$  on a Hilbert space to ultra-weakly continuous completely contractive representations of  $H^\infty(E)$  on the same Hilbert space. Our work extends the classical Sz.-Nagy - Foiaş functional calculus and more recent work by Davidson, Li and Pitts on the representation theory of Popescu's noncommutative disc algebra.

---

\*Supported in part by a grant from the U.S.-Israel Binational Science Foundation.

†Supported in part by the U.S.-Israel Binational Science Foundation and by the Lowen-gart Research Fund.

# 1 Introduction

Suppose  $\rho$  is a contractive representation of the disc algebra  $A(\mathbb{D})$  on a Hilbert space  $H$ , i.e., suppose  $\|\rho(f)\|_{B(H)} \leq \|f\|_\infty$ , where  $\|\cdot\|_{B(H)}$  is the operator norm on the space of bounded operators on  $H$ ,  $B(H)$ , and where  $\|\cdot\|_\infty$  is the sup norm on  $A(\mathbb{D})$ , taken over  $\overline{\mathbb{D}}$ . (By the maximum modulus principle, the supremum needs only to be evaluated over the circle,  $\mathbb{T}$ .) Then  $\rho$  is completely determined by its value at the identity function  $z$  in  $A(\mathbb{D})$ ,  $T := \rho(z)$ . Of course  $T$  is a contraction operator in  $B(H)$ . On the other hand, given a contraction operator  $T$  in  $B(H)$ , then von Neumann's inequality guarantees that there is a unique contractive representation  $\rho$  of  $A(\mathbb{D})$  in  $B(H)$  such that  $T = \rho(z)$ . A natural question arises: When does  $\rho$  extend to a representation of  $H^\infty(\mathbb{T})$  in  $B(H)$  that is continuous with respect to the weak-\* topology on  $H^\infty(\mathbb{T})$  and the weak-\* topology on  $B(H)$ ? (We follow the convention of calling the weak-\* topology on  $B(H)$  the ultra-weak topology.) Thanks to the Sz.-Nagy - Foiaş functional calculus [26], a neat succinct answer may be given in terms of  $T$ , viz.,  $\rho$  admits such an extension if and only if the unitary part of  $T$  is absolutely continuous. In a bit more detail, recall that an arbitrary contraction operator  $T$  on a Hilbert space  $H$  decomposes uniquely into the direct sum  $T = T_{cnu} \oplus U$ , where  $T_{cnu}$  is completely non unitary, meaning that there are no invariant subspaces for  $T_{cnu}$  on which  $T_{cnu}$  acts as a unitary operator, and where  $U$  is unitary. Thus the answer to the question is:  $\rho$  extends if and only if the spectral measure for  $U$  is absolutely continuous with respect to Lebesgue measure on  $\mathbb{T}$ . The assertion that extension is possible when  $U$  is absolutely continuous is [26, Theorem III.2.1]. The assertion that if the extension is possible, then  $U$  is absolutely continuous is essentially [26, Theorem III.2.3]. Note in particular that since the eigenspace of any eigenvalue for  $T$  of modulus one must reduce  $T$ , it follows that when  $H$  is finite dimensional  $\rho$  extends to  $H^\infty(\mathbb{T})$  as a weak-\* continuous representation if and only if the spectral radius of  $T$  is strictly less than one. And when  $\dim(H) = 1$ , we recover the well-known fact that a character of  $A(\mathbb{D})$  extends to a weak-\* continuous character of  $H^\infty(\mathbb{T})$  if and only if it comes from a point in  $\mathbb{D}$ .

We were drawn to thinking about this perspective on the Sz.-Nagy - Foiaş functional calculus by recent work we have done in the theory of tensor and Hardy algebras. Suppose  $M$  is a  $W^*$ -algebra and that  $E$  is a  $W^*$ -correspondence over  $M$  in the sense of [15]. Then, in a fashion that will be discussed more thoroughly in the next section, one can form both the

tensor algebra of  $E$ ,  $\mathcal{T}_+(E)$ , and its ultra-weak closure, the Hardy algebra of  $E$ ,  $H^\infty(E)$ . If  $M = \mathbb{C} = E$ , then  $\mathcal{T}_+(E) = A(\mathbb{D})$  and  $H^\infty(E) = H^\infty(\mathbb{T})$ . Every completely contractive representation  $\rho : \mathcal{T}_+(E) \rightarrow B(H)$  of  $\mathcal{T}_+(E)$  on a Hilbert space  $H$  with the property that  $\rho$  restricted to (the copy of)  $M$  in  $\mathcal{T}_+(E)$  is a normal representation of  $M$  on  $H$ , that we denote by  $\sigma$ , is determined uniquely by a contraction operator  $\tilde{T} : E \otimes_\sigma H \rightarrow H$  satisfying the intertwining equation

$$\tilde{T}\sigma^E \circ \varphi(\cdot) = \sigma(\cdot)\tilde{T}, \quad (1)$$

where  $\varphi$  gives the left action of  $M$  on  $E$  and where  $\sigma^E$  is the induced representation of  $\mathcal{L}(E)$  on  $E \otimes_\sigma H$  defined by the formula  $\sigma^E(a) = a \otimes I_H$ ,  $a \in \mathcal{L}(E)$ . And conversely, once  $\sigma$  is fixed, each contraction  $\tilde{T}$  satisfying this equation determines a completely contractive representation of  $\mathcal{T}_+(E)$ . We write  $\rho = T \times \sigma$ . The question we wanted to address, and which we will discuss here, is:

*What conditions must  $\tilde{T}$  satisfy so that  $T \times \sigma$  extends from  $\mathcal{T}_+(E)$  to an ultra-weakly continuous representation of  $H^\infty(E)$ ?*

It is easy to see that if  $\|\tilde{T}\| < 1$ , then  $T \times \sigma$  extends from  $\mathcal{T}_+(E)$  to an ultra-weakly continuous representation of  $H^\infty(E)$  [15, Corollary 2.14]. Thus, the question is really about operators  $\tilde{T}$  that have norm equal to one. With quite a bit more work we showed in [15, Theorem 7.3] that if  $\tilde{T}$  is *completely non-coisometric*, meaning that there is no subspace of  $H$  that is invariant under  $T \times \sigma(\mathcal{T}_+(E))^*$  to which  $\tilde{T}^*$  restricts yielding an isometry mapping to  $E \otimes_\sigma H$ , then  $T \times \sigma$  extends to an ultra-weakly continuous representation of  $H^\infty(E)$ . Thus it looks like we are well on the way to generalizing the theorems of Sz.-Nagy and Foiaş that we cited above. “All we need is a good generalization of the notion of a completely non-unitary contraction and a good generalization of an absolutely continuous unitary operator.” It turns out, however, that things are not this simple. While a natural generalization of a unitary operator is a representation  $T \times \sigma$ , where  $\tilde{T}$  is a Hilbert space isomorphism, it is not quite so clear what it means for  $\tilde{T}$  to be absolutely continuous. There is no evident notion of spectral measure for  $\tilde{T}$  in this case. Further, in the Sz.-Nagy - Foiaş theory, it is important to know about the minimal unitary extension of the minimal isometric dilation of the contraction  $T$ , i.e., it is important to know about the minimal unitary dilation of  $T$ . However, it turns out in the theory we are describing, while there is always

a unique (up to unitary equivalence) minimal isometric dilation of  $\tilde{T}$  there may be many “unitary” extensions of the isometric dilation. Straightforward definitions and results do not appear to exist.

We are not the first to ponder our basic question. We have received a lot of inspiration from two important papers: [2] and [3]. In [2], Davidson, Katsoulis and Pitts weren’t directly involved with this question, but they clearly were influenced by it. They considered the situation where  $M = \mathbb{C}$  and  $E = \mathbb{C}^d$  for a suitable  $d$ . (When  $d = \infty$ , we view  $\mathbb{C}^d$  as  $\ell^2(\mathbb{N})$ .) The tensor algebra,  $\mathcal{T}_+(\mathbb{C}^d)$ , in this case is the norm-closed algebra generated by the creation operators on the full Fock space  $\mathcal{F}(\mathbb{C}^d)$ . We fix an orthonormal basis  $\{e_i\}_{i=1}^d$  for  $\mathbb{C}^d$  and let  $L_i$  be the creation operator of tensoring with  $e_i$ . Thus  $L_i\eta = e_i \otimes \eta$  for all  $\eta \in \mathcal{F}(\mathbb{C}^d)$ . Then  $\mathcal{T}_+(\mathbb{C}^d)$  is generated by the  $L_i$  and coincides with Popescu’s *noncommutative disc algebra*, denoted  $\mathfrak{A}_d$ . The weakly closed algebra generated by the  $L_i$  is denoted  $\mathcal{L}_d$  and is called the *noncommutative analytic Toeplitz algebra*. It turns out that  $\mathcal{L}_d$  is also the ultra-weak closure of  $\mathcal{T}_+(\mathbb{C}^d)$ , and so  $\mathcal{L}_d$  coincides with  $H^\infty(\mathbb{C}^d)$ . In their setting  $\mathcal{T}_+(\mathbb{C}^d)$  and  $H^\infty(\mathbb{C}^d)$  are concretely defined operator algebras since  $\mathcal{F}(\mathbb{C}^d)$  is a Hilbert space. We single out this special representation of  $\mathcal{T}_+(\mathbb{C}^d)$  and  $H^\infty(\mathbb{C}^d)$  with the notation  $\lambda$  - for left regular representation, which it is, if  $\mathcal{F}(\mathbb{C}^d)$  is identified with the  $\ell^2$ -space of the free semigroup on  $d$  generators through a choice of basis. If  $\rho = T \times \sigma$  is completely contractive representation of  $\mathcal{T}_+(\mathbb{C}^d)$  on  $H$ , then  $\sigma$  must be a multiple of the identity representation of  $\mathbb{C}$ , namely the Hilbert space dimension of  $H$ . Also,  $\tilde{T}$  is simply the row  $d$ -tuple of operators,  $(T_1, T_2, \dots, T_d)$ , where  $T_i = \rho(L_i)$ . As an operator from  $\mathbb{C}^d \otimes H$  to  $H$ ,  $\tilde{T}$  has norm at most 1; that is  $(T_1, T_2, \dots, T_d)$  a *row contraction*. The equation (1) is automatic in this case. Davidson, Katsoulis and Pitts assume in their work that  $(T_1, T_2, \dots, T_d)$  is a row isometry, i.e., that  $\tilde{T}$  is an isometry. They are interested in how  $(T_1, T_2, \dots, T_d)$  relates to  $(L_1, L_2, \dots, L_d)$ . For this purpose they let  $\mathcal{S}$  be the weakly closed subalgebra of  $B(H)$  that is generated by  $\{T_1, T_2, \dots, T_d\}$  and the identity, and they let  $\mathcal{S}_0$  be the weakly closed *ideal* in  $\mathcal{S}$  generated by  $\{T_1, T_2, \dots, T_d\}$ . Their principal result is [2, Theorem 2.6], which they call *The Structure Theorem*. It asserts that if  $N$  denotes the von Neumann algebra generated by  $\mathcal{S}$ , and if  $p$  is the largest projection in  $N$  such that  $p\mathcal{S}p$  is self-adjoint, then

1.  $Np = \bigcap_{k \geq 1} \mathcal{S}_0^k$ ;
2.  $\mathcal{S}p = Np$ , so in particular,  $p\mathcal{S}p = pNp$ ;

3.  $p^\perp H$  is invariant under  $\mathcal{S}$  and  $\mathcal{S} = Np + p^\perp \mathcal{S} p^\perp$ ; and
4. assuming  $p \neq I$ ,  $p^\perp \mathcal{S} p^\perp$  is completely isometrically isomorphic and ultra-weakly homeomorphic to  $\mathcal{L}_d$ .

Since  $\mathcal{L}_d = H^\infty(\mathbb{C}^d)$ , it follows that if  $p = 0$ , then the representation of  $\mathcal{T}_+(\mathbb{C}^d)$  determined by the tuple  $(T_1, T_2, \dots, T_d)$  extends to  $H^\infty(\mathbb{C}^d)$  as an weak-\* continuous representation of  $H^\infty(\mathbb{C}^d)$ . If  $p \neq 0$ , then the representation may still extend to  $H^\infty(\mathbb{C}^d)$ , but the matter becomes more subtle.

As the authors of [2] observe, this decomposition, is suggestive of certain aspects of absolute continuity in the setting of a single isometry. This point is taken up in [3], where Davidson, Li and Pitts say that a vector  $x$  in the Hilbert space of  $(T_1, T_2, \dots, T_d)$ ,  $H$ , is *absolutely continuous* if the vector functional on  $\mathcal{T}_+(\mathbb{C}^d)$  it determines can be represented by a vector functional on  $\mathcal{L}_d$ , i.e., if there are vectors  $\xi, \eta \in \mathcal{F}(\mathbb{C}^d)$  such that  $(\rho(a)x, x) = (\lambda(a)\xi, \eta)$  for all  $a \in \mathcal{T}_+(\mathbb{C}^d)$ . The collection of all such vectors  $x$  is denoted  $\mathcal{V}_{ac}(\rho)$ . This set is, in fact, a closed subspace of  $H$ , and the representation  $\rho$  extends to  $H^\infty(\mathbb{C}^d)$  as an ultraweakly continuous representation if and only if  $\mathcal{V}_{ac}(\rho) = H$ . One of their main results is [3, Theorem 3.4], which implies that  $\mathcal{V}_{ac}(\rho) = H$  if and only if the structure projection for the representation determined by  $(L_1 \oplus T_1, L_2 \oplus T_2, \dots, L_d \oplus T_d)$  acting on  $\mathcal{F}(\mathbb{C}^d) \oplus H$  is zero.

A central role is played in [3] by the operators that intertwine  $\lambda$  and  $\rho$ , i.e., operators  $X : \mathcal{F}(\mathbb{C}^d) \rightarrow H$  that satisfy the equation  $\rho(a)X = X\lambda(a)$ , for all  $a \in \mathcal{T}_+(\mathbb{C}^d)$ . Theorem 2.7 of [3] shows that  $\mathcal{V}_{ac}(\rho)$  is the union of the ranges of the  $X$ 's that intertwine  $\lambda$  and  $\rho$ . In an aside [3, Remark 2.12], the authors note that Popescu [20, Theorem 3.8] has shown that if  $X$  is an intertwiner then  $XX^*$  is a nonnegative operator on  $H$  that satisfies the two conditions:

$$\Phi(XX^*) \leq XX^* \tag{2}$$

and

$$\Phi^k(XX^*) \rightarrow 0 \tag{3}$$

in the strong operator topology, where  $\Phi(Q) := \tilde{T}Q\tilde{T}^*$ , and conversely every nonnegative operator  $Q$  on  $H$  that satisfies (2) and (3) can be factored as  $Q = XX^*$ , where  $X$  is an intertwiner.

After contemplating this connection between [3] and [20], we realized that there is a very tight connection among all the various constructs we have discussed and that they all can be generalized to our setting of tensor and Hardy algebras associated to a  $W^*$ -correspondence. This is what we

do here. In the next section, we draw together a number of facts that we will use in the sequel. Most are known from the literature. In Section (3) we develop the notion of absolute continuity first for isometric representations of  $\mathcal{T}_+(E)$ , i.e. for representations  $\rho = T \times \sigma$  where  $\tilde{T}$  is an isometry. In Section 4, we study absolute continuity in the context of an arbitrary completely contractive representation  $\rho$ . Here we show that  $\rho$  extends from  $\mathcal{T}_+(E)$  to an ultraweakly continuous completely contractive representation of  $H^\infty(E)$  if and only if  $\rho$  is absolutely continuous, i.e., if and only if  $\mathcal{V}_{ac}(\rho) = H$ . It turns out that the absolutely continuous subspace  $\mathcal{V}_{ac}(\rho)$  is really an artifact of the completely positive map attached to  $\tilde{T}$ . This fact, coupled to our work in [16], which shows that every completely positive map on a von Neumann algebra gives rise to a  $W^*$ -correspondence and a representation of it, enables us to formulate a notion of absolute continuity for an arbitrary completely positive map. This formulation is made in Section 5, where other corollaries of Sections 3 and 4 are drawn. Section 6 is something of an interlude, where we deal with an issue that does not arise in [2, 3]. The analogue of the representation  $\lambda$  in our theory is a representation of  $\mathcal{T}_+(E)$  that is induced from a representation of  $M$  in the sense of Rieffel [21, 22]. Owing to the possibility that the center of  $M$  is non-trivial, an induced representation need not be faithful. This fact creates a number of technical problems for us with which we deal in Section 6. The final section, Section 7, is devoted to our generalization of the Structure Theorem of Davidson, Katsoulis and Pitts [2, Theorem 2.6] and to its connection with the notion of absolute continuity.

## 2 Background and Preliminaries

It will be helpful to have at our disposal a number of facts developed in the literature. Our presentation is only a survey, and a little discontinuous. Certainly, it is not comprehensive, but we have given labels to paragraphs for easy reference in the body of the paper.

**2.1** Throughout this paper,  $M$  will denote a fixed  $W^*$ -algebra. We do not preclude the possibility that  $M$  may be finite dimensional. Indeed, as we have indicated, the situation when  $M = \mathbb{C}$  can be very interesting. However, we want to think of  $M$  abstractly, as a  $C^*$ -algebra that is a dual space, without regard to any Hilbert space on which  $M$  might be represented. We will reserve the term “von Neumann algebra” for a concretely represented  $W^*$ -algebra. The weak-\* topology on a  $W^*$ -algebra or on any of its weak-\*

closed subspaces will be referred to as the *ultra-weak* topology. If  $S$  is a subset of a  $W^*$ -algebra, we shall write  $\overline{S}^{u-w}$  for its ultra-weak closure.

To eliminate unnecessary technicalities we shall always assume  $M$  is  $\sigma$ -finite in the sense that every family of mutually orthogonal projections in  $M$  is countable. Alternatively, to say  $M$  is  $\sigma$ -finite is to say that  $M$  has a faithful normal representation on a separable Hilbert space. So, unless explicitly indicated otherwise, every Hilbert space we consider will be assumed to be separable.

**2.2** In addition,  $E$  will denote a  $W^*$ -correspondence over  $M$  in the sense of [15]. This means first that  $E$  is a (right) Hilbert  $C^*$ -module over  $M$  that is self-dual in the sense that each (right) module map  $\Phi$  from  $E$  into  $M$  is induced by a vector in  $E$ , i.e., there is an  $\eta \in E$  such that  $\Phi(\xi) = \langle \eta, \xi \rangle$ , for all  $\xi \in E$ . Our basic reference for Hilbert  $C^*$ - and  $W^*$ -modules is [13]. It is shown in [13, Proposition 3.3.4] that when  $E$  is a self-dual Hilbert module over a  $W^*$ -algebra  $M$ , then  $E$  must be a dual space. In fact, it may be viewed as an ultra-weakly closed subspace of a  $W^*$ -algebra. Further, every continuous module map on  $E$  is adjointable [13, Corollary 3.3.2] and the algebra  $\mathcal{L}(E)$  consisting of all continuous module maps on  $E$  is a  $W^*$ -algebra [13, Proposition 3.3.4]. To say that  $E$  is a  $W^*$ -correspondence over  $M$  is to say, then, that  $E$  is a self-dual Hilbert module over  $M$  and that there is an ultra-weakly continuous  $*$ -representation  $\varphi : M \rightarrow \mathcal{L}(E)$  such that  $E$  becomes a bimodule over  $M$  where the left action of  $M$  is determined by  $\varphi$ ,  $a \cdot \xi = \varphi(a)\xi$ . We shall assume that  $E$  is *essential* or *non-degenerate* as a left  $M$ -module. This is the same as assuming that  $\varphi$  is unital. We also shall assume that our  $W^*$ -correspondences are countably generated as self-dual Hilbert modules over their coefficient algebras. This is equivalent to assuming that  $\mathcal{L}(E)$  is  $\sigma$ -finite.

**2.3** In this paper, we will be studying objects of various kinds, algebras,  $*$ -algebras, modules, etc. and we will be considering various types of linear maps of such objects to spaces of bounded operators on Hilbert spaces. We will be especially interested in spaces of intertwining operators between such maps. For this purpose, we introduce the following notation. Suppose  $\mathcal{X}$  is an object of one of the various kinds we are considering in this paper, e.g., an algebra, a bimodule, etc., and suppose that for  $i = 1, 2$ ,  $\rho_i : \mathcal{X} \rightarrow B(H_i)$  is a map of  $\mathcal{X}$  to bounded linear operators on the Hilbert space  $H_i$ . Then we write  $\mathcal{I}(\rho_1, \rho_2)$  for the space of all bounded linear operators  $X : H_1 \rightarrow H_2$  such that  $X\rho_1(\xi) = \rho_2(\xi)X$  for all  $\xi \in \mathcal{X}$ . That is,  $\mathcal{I}(\rho_1, \rho_2)$  is the *intertwining*

space or the space of all *intertwiners* of  $\rho_1$  and  $\rho_2$ . If  $\mathcal{X}$  is a  $C^*$ -algebra and  $\rho_1$  and  $\rho_2$  are  $C^*$ -representations, then  $\mathcal{I}(\rho_1, \rho_2)$  has the structure of a Hilbert  $W^*$ -module over the commutant of  $\rho_1(\mathcal{X})$ ,  $\rho_1(\mathcal{X})'$ . The  $\rho_1(\mathcal{X})'$ -valued inner product on  $\mathcal{I}(\rho_1, \rho_2)$  is given by the formula  $\langle X, Y \rangle := X^*Y$  and the right action of  $\rho(E)'$  on  $\mathcal{I}(\rho_1, \rho_2)$  is given by the formula  $X \cdot a := Xa$ ,  $X \in \mathcal{I}(\rho_1, \rho_2)$ ,  $a \in \rho(\mathcal{X})'$ . Of course, the product,  $Xa$ , is just the composition of operators. It is quite clear that  $\mathcal{I}(\rho_1, \rho_2)$  with these operations is a Hilbert  $C^*$ -module over  $\rho(\mathcal{X})'$ , but what makes it a self-dual Hilbert module is the fact that it is ultra-weakly closed in  $B(H_1, H_2)$ . See [13, Theorem 3.5.1].

If  $\mathcal{X}$  is a  $C^*$ -algebra and  $\rho_1$  and  $\rho_2$  are two  $C^*$ -representations, then it is clear that if  $\mathcal{I}(\rho_1, \rho_2)^*$  denotes the space  $\{X^* \mid X \in \mathcal{I}(\rho_1, \rho_2)\}$ , then  $\mathcal{I}(\rho_1, \rho_2)^* = \mathcal{I}(\rho_2, \rho_1)$ . Consequently, either  $\mathcal{I}(\rho_1, \rho_2)$  and  $\mathcal{I}(\rho_2, \rho_1)$  are both nonzero or both are zero (in the latter case,  $\rho_1$  and  $\rho_2$  are called *disjoint*.) However, there are situations that we shall encounter (see Remark 4.9.) where  $\mathcal{X}$  is an operator algebra and the  $\rho_i$  are completely contractive representations of  $\mathcal{X}$  with the property  $\mathcal{I}(\rho_1, \rho_2)$  is nonzero, but  $\mathcal{I}(\rho_2, \rho_1) = \{0\}$ . Thus, in the non-self-adjoint setting, one has to take extra care when manipulating intertwining spaces.

**2.4** The concept of a  $W^*$ -correspondence over a  $W^*$ -algebra is really a special case of the very useful more general notion of a  $W^*$ -correspondence from one  $W^*$ -algebra to another. Specifically, if  $M_1$  and  $M_2$  are  $W^*$ -algebras, a  $W^*$ -correspondence from  $M_1$  to  $M_2$  is a self-dual Hilbert module  $F$  over  $M_2$  endowed with a normal representation  $\varphi : M_1 \rightarrow \mathcal{L}(F)$ . We always assume that  $\varphi$  is unital. In particular, every normal representation of  $M$  on a Hilbert space  $H$  makes  $H$  a  $W^*$ -correspondence from  $M$  to  $\mathbb{C}$ , and conversely, a  $W^*$ -correspondence from  $M$  to  $\mathbb{C}$  is just a normal representation of  $M$  on a Hilbert space. Also, observe that if  $\rho_1$  and  $\rho_2$  are two representations of a  $C^*$ -algebra  $A$  on Hilbert spaces  $H_1$  and  $H_2$ , respectively, then  $\mathcal{I}(\rho_1, \rho_2)$  is a  $W^*$ -correspondence from  $\rho_2(A)'$  to  $\rho_1(A)'$ . We know already that  $\mathcal{I}(\rho_1, \rho_2)$  is a self-dual Hilbert  $C^*$ -module over  $\rho_1(A)'$ . The left action of  $\rho_2(A)'$  on  $\mathcal{I}(\rho_1, \rho_2)$  is given by composition of maps:  $b \cdot X := bX$  for all  $b \in \rho_2(A)'$  and all  $X \in \mathcal{I}(\rho_1, \rho_2)$ .

Correspondences can be “composed” through the process of balanced tensor product, but a little care must be taken. That is, if  $F$  is a  $W^*$ -correspondence from  $M_1$  to  $M_2$  and if  $G$  is a  $W^*$ -correspondence from  $M_2$  to  $M_3$  then one can form their balanced  $C^*$ -tensor product, balanced over  $M_2$ , but in general it won’t be a  $W^*$ -correspondence from  $M_1$  to  $M_3$ . So for us, the tensor product  $F \otimes_{M_2} G$ , balanced over  $M_2$ , will be the unique



self-dual completion of the balanced  $C^*$ -tensor product of  $F$  and  $G$ . (See [13, Theorem 3.2.1].) It will be a  $W^*$ -correspondence from  $M_1$  to  $M_3$ . As an example, one can easily check that  $\mathcal{I}(\rho_2, \rho_3) \otimes_{\rho_2(A)'} \mathcal{I}(\rho_1, \rho_2)$  is naturally isomorphic  $\mathcal{I}(\rho_1, \rho_3)$ , where the  $\rho_i$  are all  $C^*$ -representations of a  $C^*$ -algebra  $A$ ; the isomorphism sends  $X \otimes Y$  to  $XY$ , where  $XY$  is just the ordinary product of  $X$  and  $Y$ .

**2.5** As a special case of the composition of  $W^*$ -correspondences, we find the notion of *induced representation* in the sense of Rieffel [21, 22]. If  $F$  is a self-dual Hilbert  $C^*$ -module over the  $W^*$ -algebra  $M$ , then *inter alia*  $F$  is a  $W^*$ -correspondence from  $\mathcal{L}(F)$  to  $M$ . So if  $\sigma$  is a normal representation of  $M$  on a Hilbert space  $H$ , then  $\sigma$  makes  $H$  a  $W^*$ -correspondence from  $M$  to  $\mathbb{C}$ . Thus we get a correspondence  $F \otimes_M H$  from  $\mathcal{L}(F)$  to  $\mathbb{C}$ . If we want to think of  $F \otimes_M H$  in terms of representations, then the normal representation of  $\mathcal{L}(F)$  associated with  $F \otimes_M \mathbb{C}$  is denoted  $\sigma^F$  and is called the *representation of  $\mathcal{L}(F)$  induced by  $\sigma$* . Evidently,  $\sigma^F$  is given by the formula  $\sigma^F(T)(\xi \otimes h) = (T\xi) \otimes h$ . We will usually write  $F \otimes_M H$  as  $F \otimes_\sigma H$ . It is a consequence of [22, Theorem 6.23] that the commutant of  $\sigma^F(\mathcal{L}(F))$  is  $I_F \otimes \sigma(M)'$ .

**2.6** Putting together the structures we have discussed so far, we come to a central concept to our theory. Returning to our correspondence  $E$  over  $M$ , let  $\sigma$  be a normal representation of  $M$  on a Hilbert space  $H$ . Form the induced representation  $\sigma^E$  of  $\mathcal{L}(E)$  and form the normal representation of  $M$ ,  $\sigma^E \circ \varphi$ , which acts on  $E \otimes_\sigma H$ . (Recall that the left action of  $M$  on  $E$  is given by the normal representation  $\varphi : M \rightarrow \mathcal{L}(E)$ .) We define  $E^\sigma$  to be  $\mathcal{I}(\sigma, \sigma^E \circ \varphi)$  and we call  $E^\sigma$  the  $\sigma$ -dual of  $E$ . This is a  $W^*$ -correspondence over  $\sigma(M)'$ . The bimodule actions are given by the formula

$$a \cdot X \cdot b := (I_E \otimes a)Xb, \quad a, b \in \sigma(M)', \quad X \in \mathcal{I}(\sigma, \sigma \circ \varphi).$$

**2.7** Along with  $E$ , we may form the  $(W^*)$ -tensor powers of  $E$ ,  $E^{\otimes n}$ . They will be understood to be the self-dual completions of the  $C^*$ -tensor powers of  $E$ . Likewise, the Fock space over  $E$ ,  $\mathcal{F}(E)$ , will be the self-dual completion of the Hilbert  $C^*$ -module direct sum of the  $E^{\otimes n}$ :

$$\mathcal{F}(E) = M \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \dots$$

We view  $\mathcal{F}(E)$  as a  $W^*$ -correspondence over  $M$ , where the left and right actions of  $M$  are the obvious ones, i.e., the diagonal actions, and we shall write  $\varphi_\infty$  for the left diagonal action of  $M$ .

For  $\xi \in E$ , we shall write  $T_\xi$  for the so-called *creation operator* on  $\mathcal{F}(E)$  defined by the formula  $T_\xi \eta = \xi \otimes \eta$ ,  $\eta \in \mathcal{F}(E)$ . It is easy to see that  $T_\xi$  is in  $\mathcal{L}(\mathcal{F}(E))$  with norm  $\|\xi\|$ , and that  $T_\xi^*$  annihilates  $M$ , as a summand of  $\mathcal{F}(E)$ , while on elements of the form  $\xi \otimes \eta$ ,  $\xi \in E$ ,  $\eta \in \mathcal{F}(E)$ , it is given by the formula

$$T_\xi^*(\zeta \otimes \eta) := \varphi_\infty(\langle \xi, \zeta \rangle) \eta.$$

**Definition 2.1** *If  $E$  is a  $W^*$ -correspondence over a  $W^*$ -algebra  $M$ , then the tensor algebra of  $E$ , denoted  $\mathcal{T}_+(E)$ , is defined to be the norm-closed subalgebra of  $\mathcal{L}(\mathcal{F}(E))$  generated by  $\varphi_\infty(M)$  and  $\{T_\xi \mid \xi \in E\}$ . The Hardy algebra of  $E$ , denoted  $H^\infty(E)$ , is defined to be the ultra-weak closure in  $\mathcal{L}(\mathcal{F}(E))$  of  $\mathcal{T}_+(E)$ .*

**2.8** Plenty of examples are given in [15] and discussed in detail there. More will be given below, but we now want to describe some properties of the representation theory of  $\mathcal{T}_+(E)$  and  $H^\infty(E)$  that we shall use. Details for what we describe are presented in Section 2 of [15]. If  $\rho$  is a completely contractive representation of  $\mathcal{T}_+(E)$  on a Hilbert space  $H$ , then  $\sigma := \rho \circ \varphi_\infty$  is a  $C^*$ -representation of  $M$  on  $H$ . We shall consider only those completely contractive representations of  $\mathcal{T}_+(E)$  with the property that  $\rho \circ \varphi_\infty$  is an ultra-weakly continuous representation of  $M$ . This is not a significant restriction. In particular, it is not a restriction at all, if  $H$  is assumed to be separable, since every  $C^*$ -representation of a  $\sigma$ -finite  $W^*$ -algebra on a separable Hilbert space is automatically ultra-weakly continuous [27, Theorem V.5.1].

In addition to the representation  $\sigma$  of  $M$ ,  $\rho$  defines a bimodule map  $T$  from  $E$  to  $B(H)$  by the formula

$$T(\xi) := \rho(T_\xi).$$

To say that  $T(\cdot)$  is a bimodule map means simply that  $T(\varphi(a)\xi b) = \sigma(a)T(\xi)\sigma(b)$  for all  $a, b \in M$  and for all  $\xi \in E$ . The assumption that  $\rho$  is completely contractive guarantees that  $T$  is completely contractive with respect to the unique operator space structure on  $E$  that arises from viewing  $E$  as a corner of its linking algebra. On the other hand, the complete contractivity of  $T$  is equivalent to the assertion that the linear map  $\tilde{T}$  defined initially on the algebraic tensor product  $E \otimes H$  to  $H$  by the formula

$$\tilde{T}(\xi \otimes h) = T(\xi)h \tag{4}$$

has norm at most one and extends to a contraction, mapping  $E \otimes_\sigma H$  to  $H$ , that satisfies the equation (1) by [16, Lemma 2.16]. That lemma, coupled with [15, Theorem 2.8], also guarantees, conversely, that if  $\tilde{T}$  is a contraction from  $E \otimes_\sigma H$  to  $H$  satisfying equation (1), then the equation (4) defines a completely contractive bimodule map that together with  $\sigma$  can be extended to a completely contractive representation of  $\mathcal{T}_+(E)$  on  $H$ . For these reasons we call the pair  $(T, \sigma)$  (or the pair  $(\tilde{T}, \sigma)$ ) a *completely contractive covariant representation* of  $(E, M)$  and we call the representation  $\rho$  the *integrated form* of  $(T, \sigma)$  and write  $\rho = T \times \sigma$ . From equation (1) we see that  $\tilde{T}^*$  lies in the space we have denoted  $E^\sigma$ . So, if we write  $\mathbb{D}(E^\sigma)$  for the open unit ball in  $E^\sigma$  and  $\overline{\mathbb{D}(E^\sigma)}$  for its norm closure, then all the completely contractive representations  $\rho$  of  $\mathcal{T}_+(E)$  such that  $\rho \circ \varphi_\infty = \sigma$  are parametrized bijectively by  $\overline{\mathbb{D}(E^{\sigma*})} = \overline{\mathbb{D}(E^\sigma)^*} = \overline{\mathbb{D}(E^\sigma)}^*$ .

**2.8.1** In the special case when  $(E, M)$  is  $(\mathbb{C}^d, \mathbb{C})$ , a representation  $\sigma$  of  $\mathbb{C}$  on a Hilbert space  $H$  is quite simple; it does the only thing it can:  $\sigma(c)h = ch$ ,  $h \in H$ , and  $c \in \mathbb{C}$ . In this setting,  $E \otimes_\sigma H$  is just the direct sum of  $d$  copies of  $H$  and  $\tilde{T}$  is simply a  $d$ -tuple of operators  $(T_1, T_2, \dots, T_d)$  such that  $\|\sum_i T_i T_i^*\| \leq 1$ , i.e.  $\tilde{T}$  is a row contraction. The map  $T$ , then, is given by the formula  $T(\xi) = \sum \xi_i T_i$ , where  $\xi = (\xi_1, \xi_2, \dots, \xi_d)^\top \in \mathbb{C}^d$ . The space  $E^\sigma$  is column space over  $B(H)$ ,  $\mathbf{C}_d(B(H))$  and  $\mathbb{D}(E^\sigma)$  is simply the unit ball in  $\mathbf{C}_d(B(H))$ .

**2.8.2** If  $(T, \sigma)$  is a completely contractive covariant representation of  $(E, M)$  on a Hilbert space  $H$ , then while it is not possible to form the powers of  $\tilde{T}$ , which maps  $E \otimes_\sigma H$  to  $H$ , we can form the *generalized* powers of  $\tilde{T}$ , which map  $E^{\otimes n} \otimes H$  to  $H$ , inductively as follows: Since  $M \otimes_\sigma H$  is isomorphic to  $H$  via the map  $a \otimes h \rightarrow \sigma(a)h$ ,  $\tilde{T}_0$  is just the identity map. Of course,  $\tilde{T}_1 = \tilde{T}$ . For  $n > 0$ ,  $\widetilde{T_{n+1}} := \widetilde{T_1(I_E \otimes \tilde{T}_n)}$ , mapping  $E^{\otimes n+1} \otimes H$  to  $H$ . This sequence of maps satisfies a semigroup-like property

$$\widetilde{T_{n+m}} = \widetilde{T_m(I_{E^{\otimes m}} \otimes \tilde{T}_n)} = \widetilde{T_n(I_{E^{\otimes n}} \otimes \tilde{T}_m)}, \quad (5)$$

where we identify  $E^{\otimes m} \otimes (E^{\otimes n} \otimes_\sigma H)$  and  $E^{\otimes n} \otimes (E^{\otimes m} \otimes_\sigma H)$  with  $E^{\otimes(n+m)} \otimes_\sigma H$  [17, Section 2]. Since the maps  $\widetilde{T_n}$  are all contractions, they may be used to promote  $(T, \sigma)$  to a completely contractive covariant representation  $(T_n, \sigma)$  of  $E^{\otimes n}$  on  $H$ , simply by setting

$$T_n(\xi_1, \xi_2, \dots, \xi_n)h := T(\xi_1)T(\xi_2) \cdots T(\xi_n)h = \widetilde{T_n}(\xi_1 \otimes \xi_2 \otimes \cdots \otimes h).$$

**2.8.3** If  $(T, \sigma)$  is a completely contractive covariant representation of  $(E, M)$  on a Hilbert space  $H$ , then  $(T, \sigma)$  induces a completely positive map  $\Phi_T$  on  $\sigma(M)'$  defined by the formula

$$\Phi_T(a) := \widetilde{T}(I_E \otimes a)\widetilde{T}^* \quad a \in \sigma(M)'. \quad (6)$$

Indeed,  $\Phi_T$  is clearly a completely positive map from  $\sigma(M)'$  into  $B(H)$ , since  $a \rightarrow I_E \otimes a$  is faithful normal representation of  $\sigma(M)'$  (onto the commutant of  $\sigma^E(\mathcal{L}(E))$ ), as we have noted earlier, and  $\widetilde{T}$  is a bounded linear map from  $E \otimes_\sigma H$  to  $H$ . To see that its range is contained in  $\sigma(M)'$ , simply note that  $\widetilde{T}\sigma^E \circ \varphi = \sigma\widetilde{T}$ . So, if  $a \in \sigma(M)'$  and if  $b \in M$ , then  $\sigma(b)\Phi_T(a) = \sigma(b)\widetilde{T}(I_E \otimes a)\widetilde{T}^* = \widetilde{T}\sigma^E \circ \varphi(b)(I_E \otimes a)\widetilde{T}^* = \widetilde{T}(I_E \otimes a)\sigma^E \circ \varphi(b)\widetilde{T}^* = \widetilde{T}(I_E \otimes a)\widetilde{T}^*\sigma(b) = \Phi_T(a)\sigma(b)$ , which shows that the range of  $\Phi_T$  is contained in  $\sigma(M)'$ . Furthermore, we see that

$$\Phi_T^n(a) = \widetilde{T}_n(I_{E^{\otimes n}} \otimes a)\widetilde{T}_n^* \quad (7)$$

for all  $n$ . This is an immediate application of paragraph 2.8.2 (see [16, Theorem 3.9] for details.)

**2.9** An important tool used in the analysis of  $\mathcal{T}_+(E)$  and  $H^\infty(E)$  is the “spectral theory of the gauge automorphism group”. What we need is developed in detail in [15, Section 2]. We merely recall the essentials that we will use. Let  $P_n$  denote the projection of  $\mathcal{F}(E)$  onto  $E^{\otimes n}$ . Then  $P_n \in \mathcal{L}(\mathcal{F}(E))$  and the series

$$W_t := \sum_{n=0}^{\infty} e^{int} P_n$$

converges in the ultra-weak topology on  $\mathcal{L}(\mathcal{F}(E))$ . The family  $\{W_t\}_{t \in \mathbb{R}}$  is an ultra-weakly continuous,  $2\pi$ -periodic unitary representation of  $\mathbb{R}$  in  $\mathcal{L}(\mathcal{F}(E))$ . Further, if  $\{\gamma_t\}_{t \in \mathbb{R}}$  is defined by the formula  $\gamma_t = \text{Ad}(W_t)$ , then  $\{\gamma_t\}_{t \in \mathbb{R}}$  is an ultra-weakly continuous group of  $*$ -automorphisms of  $\mathcal{L}(\mathcal{F}(E))$  that leaves invariant  $\mathcal{T}_+(E)$  and  $H^\infty(E)$ . Indeed, the subalgebra of  $H^\infty(E)$  fixed by  $\{\gamma_t\}_{t \in \mathbb{R}}$  is  $\varphi_\infty(M)$  and  $\gamma_t(T_\xi) = e^{-it}T_\xi$ ,  $\xi \in E$ . Associated with  $\{\gamma_t\}_{t \in \mathbb{R}}$  we have the “Fourier coefficient operators”  $\{\Phi_j\}_{j \in \mathbb{Z}}$  on  $\mathcal{L}(\mathcal{F}(E))$ , which are defined by the formula

$$\Phi_j(a) := \frac{1}{2\pi} \int_0^{2\pi} e^{-int} \gamma_t(a) dt, \quad a \in \mathcal{L}(\mathcal{F}(E)), \quad (8)$$

where the integral converges in the ultra-weak topology. An alternate formula for  $\Phi_j$  is

$$\Phi_j(a) = \sum_{k \in \mathbb{Z}} P_{k+j} a P_k.$$

Each  $\Phi_j$  leaves  $H^\infty(E)$  invariant and, in particular,  $\Phi_j(T_{\xi_1} T_{\xi_2} \cdots T_{\xi_n}) = T_{\xi_1} T_{\xi_2} \cdots T_{\xi_n}$  if and only if  $n = j$  and zero otherwise. Associated with the  $\Phi_j$  are the “arithmetic mean operators”  $\{\Sigma_k\}_{k \geq 1}$  that are defined by the formula

$$\Sigma_k(a) := \sum_{|j| < k} \left(1 - \frac{|j|}{k}\right) \Phi_j(a),$$

$a \in \mathcal{L}(\mathcal{F}(E))$ . For  $a \in \mathcal{L}(\mathcal{F}(E))$ ,  $\lim_{k \rightarrow \infty} \Sigma_k(a) = a$ , where the limit is taken in the ultra-weak topology.

**2.10** A completely contractive covariant representation  $(T, \sigma)$  of  $(E, M)$  on a Hilbert space  $H$  is called *isometric* (resp. *fully coisometric*) in case  $\tilde{T} : E \otimes H \rightarrow H$  is an isometry (resp. a coisometry). It is not difficult to see that a completely contractive covariant representation  $(T, \sigma)$  is isometric if and only if its integrated form  $T \times \sigma$  is a completely isometric representation. Further, this happens if and only if  $T \times \sigma$  is the restriction to  $\mathcal{T}_+(E)$  of a  $C^*$ -representation of the  $C^*$ -subalgebra  $\mathcal{T}(E)$  of  $\mathcal{L}(\mathcal{F}(E))$  generated by  $\mathcal{T}_+(E)$ . This  $C^*$ -algebra is called the *Toeplitz algebra* of  $E$ .

A special kind of isometric covariant representations are constructed as follows. Let  $\pi_0 : M \rightarrow B(H_0)$  be a normal representation of  $M$  on the Hilbert space  $H_0$ , and let  $H = \mathcal{F}(E) \otimes_{\pi_0} H_0$ . Set  $\sigma = \pi^{\mathcal{F}(E)} \otimes I_{H_0}$ , and define  $S : E \rightarrow B(H)$  by the formula  $S(\xi) = T_\xi \otimes I_{H_0}$ ,  $\xi \in E$ . Then it is immediate that  $(S, \sigma)$  is an isometric covariant representation and we say that it is *induced by*  $\pi_0$ . We also will say  $S \times \sigma$  is induced by  $\pi_0$ . In a sense that will become clear, the representation  $\pi_0$  should be viewed as a generalization of the multiplicity of a shift.

An induced isometric covariant representation has the property that  $\widetilde{S_n S_n^*} \rightarrow 0$  strongly as  $n \rightarrow \infty$  because  $\widetilde{S_n S_n^*}$  is the projection onto  $\sum_{k \geq n} E^{\otimes k} \otimes_{\pi_0} H_0$ . In general, an isometric covariant representation  $(S, \sigma)$  and its integrated form are called *pure* if  $\widetilde{S_n S_n^*} \rightarrow 0$  strongly as  $n \rightarrow \infty$ .

It is clear that induced covariant representations are analogues of shifts. Corollary 2.10 of [17] shows that every pure isometric covariant representation of  $(E, M)$  is unitarily equivalent to an isometric covariant representation that is induced by a normal representation of  $M$ . We therefore will usually say

simply that a pure isometric covariant representation *is* induced. In Theorem 2.9 of [17] we proved a generalization of the Wold decomposition theorem that asserts that every isometric covariant representation of  $(E, M)$  decomposes as the direct sum of an induced isometric covariant representation of  $(E, M)$  and an isometric representation of  $(E, M)$  that is both isometric and fully coisometric.

**2.11** We will need an analogue of a unilateral shift of infinite multiplicity.

**Lemma 2.2** *For  $i = 1, 2$ , let  $\pi_i : M \rightarrow B(H_i)$  be normal representation of  $M$  on the Hilbert space  $H_i$ , and let  $(S_i, \sigma_i)$  be the isometric covariant representation of  $(E, M)$  induced by  $\pi_i$ .*

1. *If  $X : H_1 \rightarrow H_2$  intertwines  $\pi_1$  and  $\pi_2$ , then  $I_{\mathcal{F}(E)} \otimes X$  intertwines  $(S_1, \sigma_1)$  and  $(S_2, \sigma_2)$ . In particular, if  $\pi_1$  and  $\pi_2$  are unitarily equivalent, then so are  $(S_1, \sigma_1)$  and  $(S_2, \sigma_2)$ .*
2. *If  $(S_1, \sigma_1)$  and  $(S_2, \sigma_2)$  are unitarily equivalent, then so are  $\pi_1$  and  $\pi_2$ .*

**Proof.** The first assertion is a straightforward calculation. The key point is that  $I_{\mathcal{F}(E)} \otimes X$  represents a bounded operator from  $\mathcal{F}(E) \otimes_{\pi_1} H_1$  to  $\mathcal{F}(E) \otimes_{\pi_2} H_2$  precisely because  $X$  intertwines  $\pi_1$  and  $\pi_2$ . The second assertion may be seen as follows. Let  $U$  be a Hilbert space isomorphism from  $\mathcal{F}(E) \otimes_{\pi_1} H_1$  to  $\mathcal{F}(E) \otimes_{\pi_2} H_2$  such that  $U(S_1 \times \sigma_1) = (S_2 \times \sigma_2)U$  and let  $H_0^\infty(E) = \{a \in H^\infty(E) \mid \Phi_0(a) = 0\}$ , where  $\Phi_0$  is the zero<sup>th</sup> Fourier coefficient operator (8). Then  $\overline{H_0^\infty(E)\mathcal{F}(E)}^{u-w} = \sum_{k \geq 1} E^{\otimes k}$ . It follows that for  $i = 1, 2$ ,  $\overline{(S_i \times \sigma_i)(H_0^\infty(E))(\mathcal{F}(E) \otimes_{\pi_i} H_i)} = (\sum_{k \geq 1} E^{\otimes k}) \otimes_{\pi_i} H_i$  and, since  $U(S_1 \times \sigma_1) = (S_2 \times \sigma_2)U$ , it follows also that  $U$  carries  $(\sum_{k \geq 1} E^{\otimes k}) \otimes_{\pi_1} H_1$  onto  $(\sum_{k \geq 1} E^{\otimes k}) \otimes_{\pi_2} H_2$ . Consequently,  $U$  restricts to a Hilbert space isomorphism from  $M \otimes_{\pi_1} H_1 \simeq H_1$  onto  $M \otimes_{\pi_2} H_2 \simeq H_2$  which, because  $U(S_1 \times \sigma_1) = (S_2 \times \sigma_2)U$ , must satisfy  $U\pi_1 = \pi_2 U$ .  $\square$

We shall fix, once and for all, a representation  $(S_0, \sigma_0)$  that is induced by a faithful normal representation  $\pi$  of  $M$  that has *infinite multiplicity*. That is,  $(S_0, \sigma_0)$  acts on a Hilbert space of the form  $\mathcal{F}(E) \otimes_\pi K_0$ , where  $\pi : M \rightarrow B(K_0)$  is an infinite ampliation of a faithful normal representation of  $M$ . Then  $\sigma_0 := \pi^{\mathcal{F}(E)} \circ \varphi_\infty$ , while  $S_0(\xi) := T_\xi \otimes I_{K_0}$ ,  $\xi \in E$ . The following proposition will be used in a number of arguments below.

**Proposition 2.3** *The representation  $(S_0, \sigma_0)$  is unique up to unitary equivalence and every induced isometric covariant representation of  $(E, M)$  is unitarily equivalent (in a natural way) to a restriction of  $(S_0, \sigma_0)$  to a subspace of the form  $\mathcal{F}(E) \otimes_\pi \mathfrak{K}$ , where  $\mathfrak{K}$  is a subspace of  $K_0$  that reduces  $\pi$ .*

**Proof.** The uniqueness assertion is an immediate consequence of Lemma 2.2 and the structure of isomorphisms between von Neumann algebras [4, Theorem I.4.4.3]. If  $(R, \rho)$  is an induced representation acting, say, on the Hilbert space  $H$ , then there is a normal representation  $\rho_0$  of  $M$  on a Hilbert space  $H_0$  and a Hilbert space isomorphism  $W : H \rightarrow \mathcal{F}(E) \otimes_{\rho_0} H_0$  such that  $WR(\xi)W^{-1} = T_\xi \otimes I_{H_0}$  and such that  $W\rho(a)W^{-1} = \varphi_\infty(a) \otimes I_{H_0}$ ,  $\xi \in E$  and  $a \in M$ . That is,  $W(R \times \rho)W^{-1}$  is the induced representation  $\rho_0^{\mathcal{F}(E)}$  of  $\mathcal{L}(\mathcal{F}(E))$  restricted to  $\mathcal{T}_+(E)$  acting on  $\mathcal{F}(E) \otimes_{\rho_0} H_0$ . Since  $\pi$  is a faithful normal representation of  $M$  on  $K_0$  with infinite multiplicity, one knows that  $\rho_0$  is unitarily equivalent to  $\pi_0$  restricted to a reducing subspace  $\mathfrak{K}$  of  $K_0$ . It follows that  $\rho_0^{\mathcal{F}(E)}$  is unitarily equivalent  $\pi_0^{\mathcal{F}(E)}$ , which, in turn, is unitarily equivalent to the restriction of  $\pi^{\mathcal{F}(E)}$  to  $\mathcal{F}(E) \otimes_\pi \mathfrak{K}$ . Stringing the unitary equivalences together completes the proof.  $\square$

**Definition 2.4** *We shall refer to  $(S_0, \sigma_0)$  as the universal induced covariant representation of  $(E, M)$ .*

By Proposition 2.3,  $(S_0, \sigma_0)$  does not really depend on the choice of representation  $\pi$  used to define it. It will serve the purpose in our theory that the unilateral shift of infinite multiplicity serves in the structure theory of single operators on Hilbert space.

**2.12** A key tool in our theory is the following result that we proved as [15, Theorem 2.8].

**Theorem 2.5** *Let  $(T, \sigma)$  be a completely contractive covariant representation of  $(E, M)$  on a Hilbert space  $H$ . Then there is an isometric covariant representation  $(V, \rho)$  of  $(E, M)$  acting on a Hilbert space  $K$  containing  $H$  such that if  $P$  denotes the projection of  $K$  onto  $H$ , then*

1.  $P$  commutes with  $\rho(M)$  and  $\rho(a)P = \sigma(a)P$ ,  $a \in M$ , and
2. for all  $\eta \in E$ ,  $V(\eta)^*$  leaves  $H$  invariant and  $PV(\eta)P = T(\eta)P$ .

*The representation  $(V, \rho)$  may be chosen so that the smallest subspace of  $K$  that contains  $H$  is all of  $K$ . When this is done,  $(V, \rho)$  is unique up to unitary equivalence and is called **the minimal isometric dilation** of  $(T, \rho)$ .*

There is an explicit matrix form for  $(V, \rho)$  similar to the classical Schäffer matrix for the unitary dilation of a contraction operator. We will need parts of it, so we present the essentials here. For more details, see Theorem 3.3 and Corollary 5.21 in [18] and the proof of Theorem 2.18 in [16], in particular. Let  $\Delta = (I - \tilde{T}^* \tilde{T})^{1/2}$  and let  $\mathcal{D}$  be its range. Then  $\Delta$  is an operator on  $E \otimes_\sigma H$  and commutes with the representation  $\sigma^E \circ \varphi$  of  $M$ , by equation (3.6). Write  $\sigma_1$  for the restriction of  $\sigma^E \circ \varphi$  to  $\mathcal{D}$ . Form  $K = H \oplus \mathcal{F}(E) \otimes_{\sigma_1} \mathcal{D}$ , where the tensor product  $\mathcal{F}(E) \otimes_{\sigma_1} \mathcal{D}$  is balanced over  $\sigma_1$ . The representation  $\rho$  is just  $\sigma \oplus \sigma_1^{\mathcal{F}(E)} \circ \varphi_\infty$ . For  $V$ , we form  $E \otimes_\rho K$  and define  $\tilde{V} : E \otimes_\rho K \rightarrow K$  matricially as

$$\tilde{V} := \begin{bmatrix} \tilde{T} & 0 & 0 & \cdots & \\ \Delta & 0 & 0 & & \ddots \\ 0 & I & 0 & \ddots & \\ 0 & 0 & I & 0 & \ddots \\ \vdots & 0 & 0 & I & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}, \quad (9)$$

where the identity operators in this matrix must be interpreted as the operators that identify  $E \otimes_{\sigma_{n+1}} (E^{\otimes n} \otimes_{\sigma_1} \mathcal{D})$  with  $E^{\otimes(n+1)} \otimes_{\sigma_1} \mathcal{D}$ , and where, in turn,  $\sigma_{n+1} = \sigma_1^{E^{\otimes n}} \circ \varphi^{(n)}$ . (Here  $\varphi^{(n)}$  denotes the representation of  $M$  in  $\mathcal{L}(E^{\otimes n})$  given by the formula  $\varphi^{(n)}(a)(\xi_1 \otimes \xi_2 \cdots \otimes \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n$ .) Then it is easily checked that  $\tilde{V}$  is an isometry and that the associated covariant representation  $(V, \rho)$  is the minimal isometric dilation  $(T, \sigma)$ .

**2.13** Another important tool in our analysis is the *commutant lifting theorem* [18, Theorem 4.4]. The way we stated this result in [18], it may be difficult to recognize relations with the classical theorem. So we begin with a more “down to earth” statement. It is exactly what is proved in [18].

**Theorem 2.6** *Let  $(T, \sigma)$  be a completely contractive representation of  $(E, M)$  on a Hilbert space  $H$  and let  $(V, \rho)$  be the minimal isometric dilation of  $(T, \sigma)$  acting on the space  $K$  containing  $H$ . If  $X$  is an operator on  $H$  that commutes with  $(T \times \sigma)(\mathcal{T}_+(E))$  then there is an operator  $Y$  on  $K$  such that the following statements are satisfied:*

1.  $\|Y\| = \|X\|$ .
2.  $Y$  commutes with  $(V \times \rho)(\mathcal{T}_+(E))$ .



3.  $Y$  leaves  $K \ominus H$  invariant.

4.  $PY|_H = X$ , where  $P$  is the projection of  $K$  onto  $H$ .

Suppose, now, that for  $i = 1, 2$   $(T_i, \sigma_i)$  is a completely contractive covariant representation of  $(E, M)$  on a Hilbert space  $H_i$ . Then we shall write  $\mathcal{I}((T_1, \sigma_1), (T_2, \sigma_2))$  for the set of operators  $X : H_1 \rightarrow H_2$  such that  $XT_1(\xi) = T_2(\xi)X$  for all  $\xi \in E$  and  $X\sigma_1(a) = \sigma_2(a)X$  for all  $a \in M$ . That is,  $\mathcal{I}((T_1, \sigma_1), (T_2, \sigma_2)) = \mathcal{I}(T_1, T_2) \cap \mathcal{I}(\sigma_1, \sigma_2)$ . Alternatively,  $\mathcal{I}((T_1, \sigma_1), (T_2, \sigma_2)) = \mathcal{I}(T_1 \times \sigma_1, T_2 \times \sigma_2)$ .

**Theorem 2.7** *For  $i = 1, 2$ , let  $(T_i, \sigma_i)$  is a completely contractive covariant representation of  $(E, M)$  on a Hilbert space  $H_i$ , let  $(V_i, \rho_i)$  be the minimal isometric dilation of  $(T_i, \sigma_i)$  acting on the space  $K_i$ , and let  $P_i$  be the orthogonal projection of  $K_i$  onto  $H_i$ . Further, let  $\mathcal{I}((V_1, \rho_1), (V_2, \rho_2); H_1, H_2) = \{X \in \mathcal{I}((V_1, \rho_1), (V_2, \rho_2)) \mid XH_1^\perp \subseteq H_2^\perp\}$ . Then the map  $\Psi$  from  $B(K_1, K_2)$  to  $B(H_1, H_2)$  defined by the formula  $\Psi(X) = P_2XP_1$  maps  $\mathcal{I}((V_1, \rho_1), (V_2, \rho_2); H_1, H_2)$  onto  $\mathcal{I}((T_1, \sigma_1), (T_2, \sigma_2))$  and for every  $Y \in \mathcal{I}((T_1, \sigma_1), (T_2, \sigma_2))$ , there is an  $X \in \mathcal{I}((V_1, \rho_1), (V_2, \rho_2); H_1, H_2)$  with  $\|X\| = \|Y\|$ .*

Otherwise stated, the restriction of  $\Psi$  to  $\mathcal{I}((V_1, \rho_1), (V_2, \rho_2); H_1, H_2)$  is a complete quotient map from  $\mathcal{I}((V_1, \rho_1), (V_2, \rho_2); H_1, H_2)$  onto  $\mathcal{I}((T_1, \sigma_1), (T_2, \sigma_2))$ . Theorem 2.7 is not the statement of Theorem 4.4 of [18], but it is a consequence of [18, Theorem 4.4] and the following standard matrix trick: Write the sum  $(T_1, \sigma_1) \oplus (T_2, \sigma_2)$  acting on  $H_1 \oplus H_2$  matricially as 
$$\begin{bmatrix} (T_1, \sigma_1) & 0 \\ 0 & (T_2, \sigma_2) \end{bmatrix}.$$
 Then the minimal isometric dilation of  $(T_1, \sigma_1) \oplus (T_2, \sigma_2)$  is  $(V_1, \rho_1) \oplus (V_2, \rho_2)$ . Further, from the matricial perspective it is clear that the commutant of 
$$\begin{bmatrix} (T_1, \sigma_1) & 0 \\ 0 & (T_2, \sigma_2) \end{bmatrix}$$
 is

$$\begin{bmatrix} (T_1, \sigma_1)' & \mathcal{I}((T_2, \sigma_2), (T_1, \sigma_1)) \\ \mathcal{I}((T_1, \sigma_1), (T_2, \sigma_2)) & (T_2, \sigma_2)' \end{bmatrix}$$

and similarly for the commutant of 
$$\begin{bmatrix} (V_1, \rho_1) & 0 \\ 0 & (V_2, \rho_2) \end{bmatrix}.$$
 With these observations, it is easy to see how Theorem 2.7 follows from [18, Theorem 4.4].

**2.14** The following lemma will be used at several points in the text.

**Lemma 2.8** *Suppose  $\mathcal{A}$  and  $\mathcal{B}$  are ultra-weakly closed linear subspaces of  $B(H)$  and  $\psi : \mathcal{A} \rightarrow \mathcal{B}$  is a bounded linear map whose restriction to the unit ball of  $\mathcal{A}$  is continuous with respect to the weak operator topologies on  $\mathcal{A}$  and  $\mathcal{B}$ . Then  $\psi$  is continuous with respect to the ultra-weak topologies on  $\mathcal{A}$  and  $\mathcal{B}$ .*

**Proof.** Let  $f$  be in  $\mathcal{B}_*$  and consider the linear functional  $f \circ \psi$  (on  $\mathcal{A}$ ). Since the weak operator topology and the ultra-weak topology agree on any bounded ball of  $\mathcal{B}$ , they agree on  $\psi(\mathcal{A}_1)$ , where  $\mathcal{A}_1$  is the unit ball of  $\mathcal{A}$ . Consequently,  $f \circ \psi|_{\mathcal{A}_1}$  is ultra-weakly continuous. It now follows from [4, Theorem I.3.1, (ii4)  $\Rightarrow$  (ii1)] that  $f \circ \psi$  is ultra-weakly continuous. Since  $f$  was an arbitrary ultra-weakly continuous functional,  $\psi$  is continuous with respect to the ultra-weak topologies.  $\square$

### 3 Absolute Continuity and Isometric Representations

Throughout this paper we shall use the following standard notation. If  $\xi$  and  $\eta$  are vectors in a Hilbert space  $H$ , then  $\omega_{\xi, \eta}$  will denote the functional on  $B(H)$  defined by the formula  $\omega_{\xi, \eta}(X) := \langle X\xi, \eta \rangle$ ,  $X \in B(H)$ . If  $\xi = \eta$ , then we shall simply write  $\omega_\xi$  for  $\omega_{\xi, \eta}$ .

**Definition 3.1** *Let  $(T, \sigma)$  be a completely contractive covariant representation of  $(E, M)$  on a Hilbert space  $H$ .*

- (1) *A vector  $x \in H$  is said to be absolutely continuous in case the functional  $\omega_x \circ (T \times \sigma)$  on  $\mathcal{T}_+(E)$  can be written as  $\omega_{\xi, \eta} \circ (S_0 \times \sigma_0)$  for suitable vectors  $\xi$  and  $\eta$  in the Hilbert space  $\mathcal{F}(E) \otimes_\pi K_0$  of the universal representation  $(S_0, \sigma_0)$ . That is,*

$$\langle (T \times \sigma)(a)x, x \rangle = \langle (S_0 \times \sigma_0)(a)\xi, \eta \rangle = \langle (a \otimes I)\xi, \eta \rangle, \quad a \in \mathcal{T}_+(E).$$

- (2) *We write  $\mathcal{V}_{ac}$  (or  $\mathcal{V}_{ac}(T, \sigma)$ ) for the set of all absolutely continuous vectors in  $H$ .*
- (3) *We say that  $(T, \sigma)$  and  $T \times \sigma$  are absolutely continuous in case  $\mathcal{V}_{ac} = H$ .*

We will show eventually that  $\mathcal{V}_{ac}(T, \sigma)$  is a closed linear subspace of  $H$ , as one might expect. However, this will take a certain amount of preparation and development. At the moment, all we can say is that  $\mathcal{V}_{ac}(T, \sigma)$  is closed under scalar multiplication.

One of our principal goals is to show that a completely contractive representation  $T \times \sigma$  of  $\mathcal{T}_+(E)$  extends to an ultra-weakly continuous completely contractive representation of  $H^\infty(E)$  if and only if  $(T, \sigma)$  is absolutely continuous. The following remark suggests that one should think of being absolutely continuous as being a “local” phenomenon.

**Remark 3.2** *A vector  $x \in H$  is absolutely continuous if and only if the functional  $\omega_x \circ (T \times \sigma)$  extends to an ultra-weakly continuous linear functional on  $H^\infty(E)$ . To see this, first observe that any faithful normal representation of  $M$  induces a faithful representation of  $\mathcal{L}(\mathcal{F}(E))$  that is also a homeomorphism with respect to the ultra-weak topologies. Consequently,  $S_0 \times \sigma_0$  extends to a faithful representation of  $H^\infty(E)$  that is a homeomorphism between the ultra-weak topology on  $H^\infty(E)$  and the ultra-weak topology on  $B(\mathcal{F}(E) \otimes_\pi K_0)$  restricted to the range of  $S_0 \times \sigma_0$ . Further, since  $\pi$  is assumed to have infinite multiplicity, every ultra-weakly continuous linear functional on  $(S_0 \times \sigma_0)(H^\infty(E))$  is a vector functional, that is, it is of the form  $\omega_{\xi, \eta}$  for vectors  $\xi$  and  $\eta$  in  $\mathcal{F}(E) \otimes_\pi K_0$ . Consequently, if  $\omega_x \circ (T \times \sigma)$  extends to an ultra-weakly continuous linear functional on  $H^\infty(E)$ , it must be representable in the form  $\omega_{\xi, \eta} \circ (S_0 \times \sigma_0)$ , for suitable vectors  $\xi$  and  $\eta$  in  $\mathcal{F}(E) \otimes_\pi K_0$ . Thus,  $x$  is an absolutely continuous vector. On the other hand, if  $x$  is an absolutely continuous vector, then  $\omega_x \circ (T \times \sigma)$  can be written as  $\omega_{\xi, \eta} \circ (S_0 \times \sigma_0)$ , by definition, and so extends to an ultra-weakly continuous linear functional on  $H^\infty(E)$ .*

To show that a completely contractive representation of  $\mathcal{T}_+(E)$  is absolutely continuous if and only if it extends to an ultra-weakly continuous representation of  $H^\infty(E)$ , we begin by proving this for completely isometric representations. This fact and the technology used to prove it will be employed in the next subsection to obtain the general result.

**Definition 3.3** *Let  $(S, \sigma)$  be an isometric representation of  $(E, M)$  on  $H$ . A vector  $x \in H$  is said to be a wandering vector for  $(S, \sigma)$ , or for  $S \times \sigma$ , if for every  $k$  and every  $\xi \in E^{\otimes k}$ , the spaces  $\sigma(M)x$  and  $S(\xi)\sigma(M)x$  are orthogonal spaces.*

**Lemma 3.4** *Let  $(S, \sigma)$  be an isometric representation of  $(E, M)$  on the Hilbert space  $H$ . If  $x \in H$  is a wandering vector for  $(S, \sigma)$ , then the representation  $S \times \sigma$ , restricted to the closed invariant subspace of  $H$  generated by  $x$  is pure and, therefore, an induced representation.*

**Proof.** Suppose  $x$  is a wandering vector. Then the closed invariant subspace  $\mathcal{M}$  generated by  $x$  is the orthogonal sum  $\sum_{k=0}^{\infty} \oplus [S(E^{\otimes k})\sigma(M)x]$ . If we write  $H_0$  for  $[\sigma(M)x]$  and if  $\pi_0$  denotes the restriction of  $\sigma$  to  $H_0$ , then it is easy to check that  $S \times \sigma$ , restricted to  $\mathcal{M}$ , is unitarily equivalent to the representation induced by  $\pi_0$ .  $\square$

The proof of the next proposition was inspired by the proof of [2, Theorem 1.6].

**Proposition 3.5** *Let  $(S, \sigma)$  be an isometric representation on the Hilbert space  $H$  and suppose  $x \in \mathcal{V}_{ac}(S, \sigma)$ . Then*

- (i)  *$x$  lies in the closure of the subspace of  $H \oplus (\mathcal{F}(E) \otimes_{\pi} K_0)$  generated by the wandering vectors for the representation  $(S \times \sigma) \oplus (S_0 \times \sigma_0)$ .*
- (ii) *There is an  $X \in \mathcal{I}((S_0, \sigma_0), (S, \sigma))$  such that  $x$  lies in the range of  $X$ .*

**Proof.** Since  $x \in H$  is absolutely continuous for  $(S, \sigma)$ , we can find vectors  $\zeta, \eta$  in  $\mathcal{F}(E) \otimes_{\pi} K_0$  such that  $\langle (S \times \sigma)(a)x, x \rangle = \langle (S_0 \times \sigma_0)(a)\zeta, \eta \rangle$  for  $a \in \mathcal{T}_+(E)$ . Write  $\tau$  for the representation  $(S \times \sigma) \oplus (S_0 \times \sigma_0)$ , and then note that

$$\langle \tau(a)(x \oplus \zeta), (x \oplus (-\eta)) \rangle = \langle (S \times \sigma)(a)x, x \rangle - \langle (S_0 \times \sigma_0)(a)\zeta, \eta \rangle = 0 \quad (10)$$

for all  $a \in \mathcal{T}_+(E)$ . Let  $\mathcal{N}$  be the closed,  $\tau(\mathcal{T}_+(E))$ -invariant subspace generated by  $x \oplus \zeta$ ,  $[\tau(\mathcal{T}_+(E))(x \oplus \zeta)]$ , and write  $(R, \rho)$  for the isometric representation associated with the restriction of  $\tau$  to  $\mathcal{N}$ . Thus  $R(\xi) = (S(\xi) \oplus S_0(\xi))|_{\mathcal{N}} = (S(\xi) \oplus (T_{\xi} \otimes I_{K_0}))|_{\mathcal{N}}$  and  $\rho = (\sigma \oplus \sigma_0)|_{\mathcal{N}} = (\sigma \oplus (\varphi_{\infty} \otimes I_{K_0}))|_{\mathcal{N}}$ .

We shall use Corollary 2.10 of [17] to show that  $(R, \rho)$  is an induced representation. For that we must show that

$$\bigcap_{k=0}^{\infty} \overline{R(E^{\otimes k})(\mathcal{N})} = \{0\}.$$

But, since  $R(\xi_k) = S(\xi_k) \oplus (T_{\xi_k} \otimes I_K)$ , it is immediate that

$$\bigcap_{k=0}^{\infty} \overline{R(E^{\otimes k})(\mathcal{N})} \subseteq H \oplus \{0\}. \quad (11)$$

Since  $(R, \rho)$  is an isometric representation, we can apply the Wold decomposition theorem ([17, Theorem 2.9]) and write  $(R, \rho) = (R_1, \rho_1) \oplus (R_2, \rho_2)$  where  $(R_1, \rho_1)$  is an induced isometric representation and where  $(R_2, \rho_2)$  is fully coisometric. This decomposition enables us, then, to write  $x \oplus \zeta = \lambda + \mu$ , where  $\lambda$  and  $\mu$  are the projections of  $x \oplus \zeta$  into the induced and the coisometric subspaces, respectively, for  $(R, \rho)$ . (We want to emphasize that we are not claiming  $\lambda$  lies in  $H$  and  $\mu$  lies in  $\mathcal{F}(E) \otimes_\pi K_0$ ; i.e., the Wold direct sum decomposition may be different from the direct sum decomposition  $H \oplus (\mathcal{F}(E) \otimes_\pi K_0)$ .) It follows from equation (11), however, that  $\mu$  is of the form  $h \oplus 0$  for some  $h \in H$  and, thus,  $\lambda = (x - h) \oplus \zeta$ . Since  $\mu$  is orthogonal to  $\lambda$ ,  $h$  is orthogonal to  $x - h$ . Also, it follows from equation (10) that  $h \oplus 0$  is orthogonal to  $x \oplus (-\eta)$ . Thus  $h$  is orthogonal to  $x$ . Since it is also orthogonal to  $x - h$ ,  $h = 0$ , implying that  $\mu = 0$  and, thus, that the representation  $(R, \rho)$  is induced. But that implies that every vector in  $\mathcal{N}$  is in the closure of the subspace spanned by the wandering vectors of  $\tau$ . In particular,  $x \oplus \zeta$  lies there. Now note that we could have replaced  $\zeta$  and  $\eta$  by  $t\zeta$  and  $t^{-1}\eta$  for any  $t > 0$ . We would then find that  $x \oplus t\zeta$  lies in the span of the wandering vectors for every  $t > 0$ . Letting  $t \rightarrow 0$ , we find that  $x$  lies in that span, completing the proof of (i).

To prove (ii), first let  $X_0$  be the projection of  $H \oplus (\mathcal{F}(E) \otimes_\pi K_0)$  onto  $H$ , restricted to  $\mathcal{N}$ . Then  $x = X_0(x \oplus \zeta)$ . By construction,  $X_0$  intertwines  $R \times \rho$  and  $S \times \sigma$ . However, by Lemma 2.3 and the fact that  $(R, \rho)$  is induced, we find that  $R \times \rho$  is unitarily equivalent to a summand of  $S_0 \times \sigma_0$ . Taking the equivalence into account,  $X_0$  can be exchanged for an  $X$  that intertwines  $S_0 \times \sigma_0$  and  $S \times \sigma$ , and has  $x$  in its range.  $\square$

**Lemma 3.6** *Suppose  $(S, \sigma)$  is an isometric representations of  $(E, M)$  on a Hilbert space  $H$  and suppose that  $X : \mathcal{F}(E) \otimes_\pi K_0 \rightarrow H$  is an element of the intertwining space  $\mathcal{I}((S_0 \times \sigma_0), (S \times \sigma))$ . If  $\mathcal{X}$  denotes the closure of the range of  $X$ , then  $\mathcal{X}$  is invariant under  $(S, \sigma)$  and the restriction  $(R, \rho)$  of  $(S, \sigma)$  to  $\mathcal{X}$  is an isometric representation. Also,  $R \times \rho$  admits a unique extension to a representation of  $H^\infty(E)$  on  $\mathcal{X}$  that is ultra-weakly continuous and completely isometric. Consequently,*

$$\mathcal{X} = \mathcal{V}_{ac}(R, \rho) \subseteq \mathcal{V}_{ac}(S, \sigma).$$

**Proof.** The fact that  $R \times \rho$  admits such an extension is proved in [15, Lemma 7.12]. It then follows that for  $x \in \mathcal{X}$ , the functional  $\omega_x \circ (R \times \rho) = \omega_x \circ (S \times \sigma)$

extends to an ultra-weakly continuous functional on  $H^\infty(E)$ . By Remark 3.2, this proves the last statement of the lemma.  $\square$

**Theorem 3.7** *If  $(S, \sigma)$  is an isometric covariant representation of  $(E, M)$ , then*

$$\mathcal{V}_{ac}(S, \sigma) = \bigvee \{ \text{Ran}(X) \mid X \in \mathcal{I}((S_0, \sigma_0), (S, \sigma)) \},$$

*and so in particular  $\mathcal{V}_{ac}(S, \sigma)$  is a closed,  $\sigma(M)$ -invariant subspace of  $H$ .*

**Proof.** We already noted that  $\mathcal{V}_{ac}(S, \sigma)$  is closed under scalar multiplication. To see that it is closed under addition, fix  $x, y \in \mathcal{V}_{ac}(S, \sigma)$ . Then, by Proposition 3.5(ii), there are operators  $X, Y \in \mathcal{I}((S_0, \sigma_0), (S, \sigma))$  such that  $x = X(\xi)$  and  $y = Y(\eta)$  for suitable vectors  $\xi$  and  $\eta$  in  $\mathcal{F}(E) \otimes_\pi K_0$ . Since  $\pi$  has infinite multiplicity, we may assume that the initial spaces of  $X$  and  $Y$  are orthogonal and, in particular, that  $\xi$  and  $\eta$  are orthogonal. It follows, then, that if we set  $Z := X \oplus Y$ , then  $Z \in \mathcal{I}((S_0, \sigma_0), (S, \sigma))$ , and  $Z(\xi + \eta) = x + y$ . Lemma 3.6 implies that  $x + y \in \mathcal{V}_{ac}(S, \sigma)$ . But also,  $\overline{\text{Ran}(X)} \subseteq \mathcal{V}_{ac}(S, \sigma)$  for every  $X \in \mathcal{I}((S_0, \sigma_0), (S, \sigma))$ , by Lemma 3.6. Thus, it remains to show that  $\mathcal{V}_{ac}(S, \sigma)$  is closed. To this end, suppose  $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathcal{V}_{ac}(S, \sigma)$  is a sequence that converges to  $x$  in  $H$ . Then the ultra-weakly continuous linear functionals  $\omega_{x_n} \circ (S \times \sigma)$  converge in norm to  $\omega_x \circ (S \times \sigma)$ , since, in general  $\|\omega_x \circ (S \times \sigma) - \omega_y \circ (S \times \sigma)\| \leq \|x - y\|$ . But the ultra-weakly continuous linear functionals on  $H^\infty(E)$  form a norm closed subset of the dual space of  $H^\infty(E)$ . Thus  $\omega_x \circ (S \times \sigma)$  extends to an ultra-weakly continuous functional on  $H^\infty(E)$ . By Remark 3.2,  $x \in \mathcal{V}_{ac}(S, \sigma)$ .  $\square$

**Corollary 3.8** *If  $(S, \sigma)$  is an isometric representation of  $(E, M)$  on  $H$ , then*

$$\mathcal{V}_{ac}(S, \sigma) = H \cap \overline{\text{span}}\{ \text{the wandering vectors of } \rho := (S \times \sigma) \oplus (S_0 \times \sigma_0) \}$$

$$\supseteq \overline{\text{span}}\{ \text{the wandering vectors of } S \times \sigma \}.$$

**Proof.** If  $x$  is a wandering vector for  $S \times \sigma$ , then the restriction of  $(S, \sigma)$  to the smallest  $S \times \sigma$ -invariant subspace  $\mathcal{N}$  spanned by  $x$  is an induced isometric representation, by Lemma 3.4. Let  $X$  be the inclusion of  $\mathcal{N}$  into  $H$ . Then it follows from Lemma 3.6 that  $x \in \mathcal{V}_{ac}(S, \sigma)$ . Now suppose  $x \oplus \zeta \in H \oplus (\mathcal{F}(E) \otimes_\pi K_0)$  is a wandering vector for  $\rho$ . Then the same argument shows that the functional  $\omega_{x \oplus \zeta} \circ \rho$  on  $\mathcal{T}_+(E)$  extends to an ultra-weakly continuous linear functional on  $H^\infty(E)$ . The same applies

to the functional  $\omega_\zeta \circ (S_0 \times \sigma_0)$ . Thus  $\omega_x \circ (S_0 \times \sigma_0)$  extends to an ultra-weakly continuous functional on  $H^\infty(E)$ . By Proposition 3.5 and Corollary 3.7  $x \in \mathcal{V}_{ac}(S, \sigma)$ .  $\square$

**Remark 3.9** *In general, the closed linear span of the wandering vectors of  $S \times \sigma$  is a proper subspace of  $\mathcal{V}_{ac}(S, \sigma)$ . Indeed, it can be zero and yet  $\mathcal{V}_{ac}(S, \sigma)$  is the whole space. Let  $S$  be the unitary operator on  $L^2$  of the upper half of the unit circle with Lebesgue measure that is given by multiplication by the independent variable. Then  $S$  is an absolutely continuous unitary operator, but it has no wandering vectors.*

**Theorem 3.10** *For an isometric representation  $(S, \sigma)$  of  $(E, M)$  on a Hilbert space  $H$  the following assertions are equivalent:*

- (1)  $S \times \sigma$  admits an ultra-weakly continuous extension to a completely isometric representation of  $H^\infty(E)$  on  $H$ .
- (2)  $(S, \sigma)$  is absolutely continuous (i.e.,  $\mathcal{V}_{ac}(S, \sigma) = H$ ).
- (3)  $H$  is contained in the closed linear span of the wandering vectors of  $(S, \sigma) \oplus (S_0, \sigma_0)$ .

**Proof.** It is clear that (1) implies (2). The equivalence of (2) and (3) follows from Corollary 3.8. It is left to show that (2) implies (1). So assume (2) holds and for every  $X \in \mathcal{I}((S_0, \sigma_0)(S, \sigma))$ , let  $\overline{Ran(X)}$  be the closure of the range of  $X$ . It follows from the assumption that  $\mathcal{V}_{ac}(S, \sigma) = H$  and Proposition 3.5 that  $H$  is spanned by the family of subspaces  $\{\overline{Ran(X)} \mid X \in \mathcal{I}((S_0, \sigma_0)(S, \sigma))\}$  and, furthermore, the restriction of  $S \times \sigma$  to each subspace  $\overline{Ran(X)}$  in this family extends to an ultra-weakly continuous, completely isometric, representation of  $H^\infty(E)$  that we shall denote by  $(S \times \sigma)_X$ . We need to show that these “restriction representations” can be glued together to form an ultra-weakly continuous completely contractive extension of  $S \times \sigma$ .

To this end, fix an operator  $a \in H^\infty(E)$  and recall that  $\Sigma_k(a)$  denotes the  $k^{th}$ -arithmetic mean of the Taylor series of  $a$ . The  $\Sigma_k(a)$  all lie in  $\mathcal{T}_+(E)$ , satisfy the inequality  $\|\Sigma_k(a)\| \leq \|a\|$ , and converge to  $a$  in the ultra-weak topology on  $H^\infty(E)$ . Since the sequence  $(S \times \sigma)(\Sigma_k(a))$  is uniformly bounded in  $B(H)$ , it has an ultra-weak limit point in  $B(H)$ . Any two limit points must agree on each of the spaces  $\overline{Ran(X)}$  since the restrictions  $(S \times \sigma)(\Sigma_k(a))|_{\overline{Ran(X)}}$  must converge to  $(S \times \sigma)_X$ . Since the spaces

$\overline{\text{Ran}(X)}$  span  $H$ , we see that there is only one limit point  $\Theta(a)$  of the sequence  $(S \times \sigma)(\Sigma_k(a))$ . Thus the sequence  $\{(S \times \sigma)(\Sigma_k(a))\}_{k \in \mathbb{N}}$  converges ultra-weakly to  $\Theta(a)$ . Moreover, for  $x \in \overline{\text{Ran}(X)}$ ,  $\Theta(a)x = (S \times \sigma)_X(a)x$ . The same sort of reasoning shows that  $\Theta$ , so defined, is a completely isometric representation of  $H^\infty(E)$  on  $H$  that extends  $S \times \sigma$ . It remains to show that  $\Theta$  is ultra-weakly continuous. For this it suffices to show that if  $\{a_\alpha\}_{\alpha \in A}$  is a *bounded* net  $H^\infty(E)$  converging ultra-weakly in  $H^\infty(E)$  to an element  $a \in H^\infty(E)$ , then  $\{\Theta(a_\alpha)\}_{\alpha \in A}$  converges ultra-weakly to  $\Theta(a)$ . Since  $\Theta$  is continuous,  $\{\Theta(a_\alpha)\}_{\alpha \in A}$  is a bounded net and so we need only show that it converges *weakly* to  $\Theta(a)$ . But for any  $x \in H$ , we can find an  $X \in \mathcal{I}((S_0, \sigma_0)(S, \sigma))$  so that  $x \in \text{Ran}(X)$  by Lemma 3.5. We conclude, then, that  $\omega_x \circ \Theta(a_\alpha) = \omega_x \circ (S \times \sigma)_X(a_\alpha) \rightarrow \omega_x \circ (S \times \sigma)_X(a) = \omega_x \circ \Theta(a)$ . Thus  $\Theta(a_\alpha) \rightarrow \Theta(a)$  weakly. It follows (Lemma 2.8) that  $\Theta$  is  $\sigma$ -weakly continuous on  $H^\infty(E)$ . This proves that (2) implies (1).  $\square$

## 4 Completely Contractive Representations and Completely Positive Maps

As we saw in paragraph 2.8.3, if  $(T, \sigma)$  is a completely contractive covariant representation of  $(E, M)$  on a Hilbert space  $H$ , then  $(T, \sigma)$  induces a completely positive map  $\Phi_T$  on  $\sigma(M)'$  defined by the formula

$$\Phi_T(a) := \tilde{T}(I_E \otimes a)\tilde{T}^* \quad a \in \sigma(M)'. \quad (12)$$

One of our goals is to show that the absolutely continuous subspace  $\mathcal{V}_{ac}(T, \sigma)$  can be described completely in terms of  $\Phi_T$ . We therefore want to begin by showing that given a contractive, normal, completely positive map  $\Phi$  on a  $W^*$ -algebra  $M$  and a normal representation  $\rho$  of  $M$  on a Hilbert space  $H$ , then there is a canonical way to view  $\rho \circ \Phi$  as a  $\Phi_T$  for a certain  $T$  attached to a completely contractive covariant representation  $(T, \sigma)$  of a natural correspondence over  $\rho(M)'$ . We will then prove that  $\mathcal{V}_{ac}(T, \sigma)$  is an artifact of  $\Phi$ . For this first step, it will be convenient for later use to omit the assumption that our completely positive maps are contractions, in the following theorem.

**Theorem 4.1** *Given a normal completely positive map  $\Phi$  on a  $W^*$ -algebra  $M$  and a normal  $*$ -representation  $\rho$  of  $M$  on a Hilbert space  $H$ , there is a*



canonical triple  $(E, \eta, \sigma)$ , where  $E$  is a  $W^*$ -correspondence over the commutant of  $\rho(M)$ ,  $\rho(M)'$ ,  $\sigma$  is a normal  $*$ -representation of  $\rho(M)'$ , and where  $\eta$  is an element of  $E^\sigma$ , such that

$$\rho(\Phi(a)) = \eta^*(I_E \otimes \rho(a))\eta \quad (13)$$

for all  $a \in M$ . The triple  $(E, \eta, \sigma)$  is essentially unique in the following sense: If  $(E_1, \eta_1, \sigma_1)$  is another triple consisting of a  $W^*$ -correspondence  $E_1$  over  $\rho(M)'$ , a normal  $*$ -representation  $\sigma_1$  of  $\rho(M)'$  and an element  $\eta_1$  of  $E_1^{\sigma_1}$  such that  $\rho(\Phi(a)) = \eta_1^*(I_{E_1} \otimes \rho(a))\eta_1$  for all  $a \in M$ , then the kernel of  $\eta_1^*$  is of the form  $\sigma^{E_1}(q_1)E_1 \otimes H$  for a projection  $q_1 \in \mathcal{L}(E_1)$  and there is an adjointable, surjective, bi-module map  $W : E_1 \rightarrow E$  such that  $\eta_1^* = \eta^*(W \otimes I)$  and such that  $W^*W = I_{E_1} - q_1$ . Further,  $\sigma_1$  differs from  $\sigma$  by an automorphism of  $\rho(M)'$ , i.e.,  $\sigma_1 = \sigma \circ \alpha$  for a suitable automorphism  $\alpha$  of  $\rho(M)'$ .

**Proof.** We present an outline of the existence of  $(E, \eta, \sigma)$  since parts of the argument will be useful later. The details may be found in [16]. The uniqueness is proved in [24, Theorem 2.6] and we omit those details here. First, recall Stinespring's dilation theorem [25] and Arveson's proof of it [1]. Form the Stinespring space  $M \otimes_{\rho \circ \Phi} H$ , which is the completion of the algebraic tensor product  $M \odot H$  in the inner product derived from the formula

$$\langle a \otimes h, b \otimes k \rangle := \langle h, \rho \circ \Phi(a^*b)k \rangle,$$

and view  $M$  as acting on  $M \otimes_{\rho \circ \Phi} H$  through the Stinespring representation,  $\pi: \pi(a)(b \otimes h) = ab \otimes h$ . Let  $V$  be the map from  $H$  to  $M \otimes_{\rho \circ \Phi} H$  defined by the formula  $Vh = I \otimes h$ . Then the equation

$$\begin{aligned} \langle \pi(a)Vh, Vk \rangle &= \langle (a \otimes h), 1 \otimes k \rangle \\ &= \langle h, \rho \circ \Phi(a^*)k \rangle \\ &= \langle \rho \circ \Phi(a)h, k \rangle, \end{aligned} \quad (14)$$

which is valid for all  $a \in M$  and  $h, k \in M$ , shows that  $V$  is bounded, with norm  $\|\Phi(I)\|^{\frac{1}{2}}$ , and that

$$V^*\pi(a)V = \rho(\Phi(a)) \quad (15)$$

for all  $a \in M$ .

Then let  $E$  be the intertwining space  $\mathcal{I}(\rho, \pi)$ , i.e.,  $\mathcal{I}(\rho, \pi) = \{X \in B(H, M \otimes_{\rho \circ \Phi} H) \mid X\rho(a) = \pi(a)X, \text{ for all } a \in M\}$ . As we noted in paragraph 2.3, this space is a  $W^*$ -correspondence from the commutant of  $\pi(M)$ ,

$\pi(M)'$ , to the commutant of  $\rho(M)$ ,  $\rho(M)'$ . However, the map  $a \rightarrow I_M \otimes a$  is normal representation of  $\rho(M)'$  into the commutant of  $\pi(M)$ , and so by restriction,  $E = \mathcal{I}(\rho, \pi)$  becomes a  $W^*$ -correspondence over  $\rho(M)'$ . The bimodule structure is given by the formula

$$a \cdot X \cdot b = (I_M \otimes a)Xb,$$

$a, b \in \rho(M)'$ . We let  $\sigma$  be the identity representation of  $\rho(M)'$  on  $H$ .

To define  $\eta \in E^\sigma$ , we first observe that there is a Hilbert space isomorphism  $U : E \otimes_\sigma H \rightarrow M \otimes_{\rho \circ \Phi} H$  defined by the formula  $U(X \otimes h) := Xh$ . The fact that  $U$  is isometric is immediate from the way the  $\rho(M)'$ -valued inner product on  $E$  is defined. The fact that  $U$  is surjective is Lemma 2.10 of [16]. Further, a straightforward computation shows that  $U(I_E \otimes \rho(a))U^{-1} = \pi(a)$  for all  $a \in M$ . Indeed, if  $X \in E = \mathcal{I}(\rho, \pi)$  and if  $h \in H$ , then for  $a \in M$ ,

$$U(I_E \otimes \rho(a))(X \otimes h) = X\rho(a)h = \pi(a)Xh = \pi(a)U(X \otimes h).$$

That is,  $U(I_E \otimes \rho(\cdot)) = \pi(\cdot)U$ . Second, we note that since  $I_M \otimes \sigma(a)$  lies in  $\pi(M)'$  for all  $a \in \rho(M)'$ , a similar calculation shows that  $U(\varphi(\cdot) \otimes I_H) = (I_M \otimes \sigma(\cdot))U$ . Finally, observe from the definition of  $V$  that  $V\sigma(\cdot) = I_M \otimes \sigma(\cdot)V$ . Now set  $\eta = U^*V$ . Then  $\eta \in E^\sigma$  since for all  $a \in \rho(M)'$ ,  $\eta\sigma(a) = U^*V\sigma(a) = U^*(I_M \otimes \sigma(a))V = (\varphi(a) \otimes I_H)U^*V = (\varphi(a) \otimes I_H)\eta$ . Also,

$$\eta^*(I_E \otimes \rho(a))\eta = V^*U(I_E \otimes \rho(a))U^*V = V^*\pi(a)V = \rho \circ \Phi(a).$$

□

**Remark 4.2** *The cb-norm of any completely positive map is the norm of its value at the identity. So  $\|\Phi_\eta\|_{cb} = \|\eta^*\eta\|_{\sigma(M)'} = \|\eta\|_{E^\sigma}^2$ . Consequently,  $\Phi_\eta$  is contractive and completely positive if and only if  $\eta \in \overline{\mathbb{D}(E^\sigma)}$ . We thus see that every contractive completely positive map  $\Phi$  on a  $W^*$ -algebra can be realized in terms of a completely contractive covariant representation of the natural  $W^*$ -correspondence  $E$  we just constructed from it. We call  $E$  the Arveson-Stinespring correspondence associated to  $\Phi$  (see [16]). It depends on a choice of a representation of  $M$ , but that will only be emphasized when necessary. The ultra-weakly continuous, completely contractive covariant representation  $(T, \sigma)$  of  $(E, \rho(M)')$  such that  $\rho \circ \Phi = \Phi_\eta$ , where  $\tilde{T} = \eta^*$  is called the identity representation. The advantage of the identity representation of a completely*

positive map through equation (13) in Theorem 4.1 over the Stinespring representation, equation (15), is that one can express the powers of  $\Phi$  in terms of it as we discussed in paragraph 2.8.3. In this setting, equation 7 becomes

$$\rho(\Phi^n(a)) = \widetilde{T}_n(I_{E^{\otimes n}} \otimes \rho(a))\widetilde{T}_n^*.$$

**Example 4.3** To illustrate these constructs in a concrete example, let  $M = \ell^\infty(\{1, 2, \dots, n\})$  and let  $\sigma$  represent  $M$  on the Hilbert space  $\mathbb{C}^n$  as diagonal matrices. Thus  $\sigma(\varphi) = \text{diag}(\varphi_1, \varphi_2, \dots, \varphi_n)$ . Of course,  $\sigma(M)$  is the masa  $\mathbf{D}_n$  consisting of all diagonal and so  $\sigma(M)' = \mathbf{D}_n$ , too. Also, let  $A = (a_{ij})$  be an  $n \times n$  sub-Markov matrix. This means that the  $a_{ij}$  are all non-negative, and that for each  $i$ ,  $\sum_j a_{ij} \leq 1$ . Such a matrix determines a completely positive, contractive map  $\Phi$  on  $\mathbf{D}_n$  through the formula

$$\Phi(\underline{d}) := \text{diag}\left(\sum_j a_{1j}d_j, \sum_j a_{2j}d_j, \dots, \sum_j a_{nj}d_j\right),$$

where  $\underline{d} = (d_1, d_2, \dots, d_n)$ . We let  $\varepsilon_i$  be the diagonal matrix with zeros everywhere but in the  $i^{\text{th}}$  row and column, where it is a one, and we let  $\{e_i\}_{i=1}^n$  be the standard basis for  $\mathbb{C}^n$ . Then the vectors  $\varepsilon_i \otimes e_j$ ,  $i, j = 1, 2, \dots, n$ , span  $\sigma(M)' \otimes_{\Phi} \mathbb{C}^n$ , and an easy calculation shows that

$$\langle \varepsilon_i \otimes e_j, \varepsilon_k \otimes e_l \rangle = a_{ji}$$

if and only  $(i, j) = (k, l)$  and is zero otherwise. It follows that in  $\sigma(M)' \otimes_{\Phi} \mathbb{C}^n$ ,  $\varepsilon_i \otimes e_j$  is nonzero if and only if  $(j, i)$  lies in the support of  $A$ , which we denote by  $G^1$ . (The reason for the super script is that we are going to view  $G^1$  as the edge set of a graph. The vertex set,  $G^0$ , is  $\{1, 2, \dots, n\}$ .) The calculation just completed shows that

$$\{a_{ji}^{-\frac{1}{2}} \varepsilon_i \otimes e_j \mid (j, i) \in G^1\}$$

is an orthonormal basis for  $\sigma(M)' \otimes_{\Phi} \mathbb{C}^n = \mathbf{D}_n \otimes_{\Phi} \mathbb{C}^n$ . We let  $\lambda$  be the representation of  $\sigma(M)' = \mathbf{D}_n$  on  $\sigma(M)' \otimes_{\Phi} \mathbb{C}^n$  is given by the formula  $\lambda(\underline{d})(\varepsilon_i \otimes e_j) = d_i \varepsilon_i \otimes e_j$ . We also let  $\iota$  be the identity representation of  $\sigma(M)' = \mathbf{D}_n$  on  $\mathbb{C}^n$ . Then the Arveson-Stinespring correspondence in this case is  $E = \mathcal{I}(\iota, \lambda)$ . An operator  $X$  from  $\mathbb{C}^n$  to  $\sigma(M)' \otimes_{\Phi} \mathbb{C}^n$  is given by a matrix that we shall write  $[X((i, j), k)]_{(i, j) \in G^1, k \in \{1, 2, \dots, n\}}$ . The formula for  $X((i, j), k)$  is, of course,

$$X((i, j), k) = \langle X e_k, a_{ji}^{-\frac{1}{2}} \varepsilon_i \otimes e_j \rangle.$$

Since an  $X$  in  $E$  intertwines the identity representation of  $\sigma(M)' = \mathbf{D}_n$  on  $\mathbb{C}^n$  and  $\lambda$ , it follows from this equation that  $X((i, j), k)$  is zero unless  $i = k$ , when  $X \in E$ . Thus  $E$  may be viewed as a space of functions supported on  $G^1$ . Now for  $X$  and  $Y$  in  $E$ ,  $X^*Y(i, j) = \sum_{(k, l) \in G^1} \overline{X((k, l), i)} Y((k, l), j)$ . Since  $X((k, l), i) = 0$ , unless  $k = i$  and since  $Y((k, l), j) = 0$ , unless  $k = j$ , we see that  $X^*Y(i, j) = 0$ , unless  $i = j$ , in which case we find that  $X^*Y(i, i) = \sum_{l=1}^n \overline{X((i, l), i)} Y((i, l), i)$ . So, if  $X_{(i, j)}$ ,  $(i, j) \in G^1$ , is defined by the formula

$$X_{(i, j)}((k, l), m) = a_{ji}^{-\frac{1}{2}},$$

when  $k = m = i$  and  $l = j$ , and zero otherwise, then  $\{X_{(i, j)}\}_{(i, j) \in G^1}$  is an orthonormal basis for  $E$ . It follows, then, that  $\{X_{(i, j)} \otimes e_i\}_{(i, j) \in G^1}$  is an orthonormal basis for  $E \otimes_{\sigma(M)} \mathbb{C}^n$  (owing to the fact that  $X \otimes \underline{d}h = X \cdot \underline{d} \otimes h$  for all  $X \otimes h \in E \otimes_{\sigma(M)} \mathbb{C}^n$  and for all  $\underline{d} \in \mathbf{D}_n$ .) The map  $U : E \otimes_{\sigma(M)} \mathbb{C}^n \rightarrow \sigma(M)' \otimes_{\Phi} \mathbb{C}^n = \mathbf{D}_n \otimes_{\Phi} \mathbb{C}^n$  is given by the formula  $U(X \otimes h) = Xh$  and so, at the level of coordinates, we find that  $U(X \otimes h)(i, j) = X((i, j), i)h(i)$ , where  $(j, i)$  lies in  $G^1$ . In particular, we see that  $U(X_{(i, j)} \otimes e_i)(i, j) = X((i, j), i) = a_{ji}^{-\frac{1}{2}}$  so that  $U(X_{(i, j)} \otimes e_i) = a_{ji}^{-\frac{1}{2}}(\varepsilon_i \otimes e_j)$ . The map  $V : \mathbb{C}^n \rightarrow \sigma(M)' \otimes_{\Phi} \mathbb{C}^n = \mathbf{D}_n \otimes_{\Phi} \mathbb{C}^n$  is defined by the formula  $Vh = 1 \otimes h$ , where in this case,  $1$  denotes the identity matrix. Recapitulating an earlier calculation we see that

$$\begin{aligned} \langle V^*(\varepsilon_i \otimes e_j), e_k \rangle &= \langle \varepsilon_i \otimes e_j, V e_k \rangle \\ &= \langle \varepsilon_i \otimes e_j, 1 \otimes e_k \rangle \\ &= \langle e_j, \Phi(\varepsilon_i^*) e_k \rangle \\ &= \langle e_j, a_{ki} e_k \rangle. \end{aligned}$$

With all the pieces calculated, we see that the map  $\tilde{T} : E \otimes_{\sigma(M)} \mathbb{C}^n \rightarrow \mathbb{C}^n$  is defined on basis vectors for  $E \otimes_{\sigma(M)} \mathbb{C}^n$  by the equation

$$\begin{aligned} \tilde{T}(X_{(i, j)} \otimes e_i) &= V^* U(X_{(i, j)} \otimes e_i) \\ &= V^*(a_{ji}^{-\frac{1}{2}} \varepsilon_i \otimes e_j) \\ &= a_{ji} a_{ji}^{-\frac{1}{2}} e_j \\ &= a_{ji}^{\frac{1}{2}} e_j. \end{aligned} \tag{16}$$

We will use these calculations in later examples.

**Definition 4.4** Let  $\Phi$  be a completely positive operator on a  $W^*$ -algebra  $M$ . An element  $Q \in M$  is called a *superharmonic operator* in case  $Q \geq 0$  and

$$\Phi(Q) \leq Q. \quad (17)$$

If, in addition, the sequence  $\{\Phi^n(Q)\}_{n \in \mathbb{N}}$  converges to zero strongly, then we say that  $Q$  is a *pure superharmonic operator*. A superharmonic operator  $Q$  such that  $\Phi(Q) = Q$  is called *harmonic*.

If  $M$  is  $L^\infty(X, \mu)$  for some probability space  $(X, \mu)$ , then a superharmonic operator is a superharmonic function in the sense of Markov processes. (See [23, Definition 2.1.1].)

**Remark 4.5** There is an analogue of the Riesz decomposition theorem for superharmonic functions, viz: If  $Q$  is a superharmonic operator for  $\Phi$ , then  $Q$  decomposes uniquely as  $Q = Q_p + Q_h$ , where  $Q_p$  is a pure superharmonic operator for  $\Phi$  and  $Q_h$  is a harmonic operator for  $\Phi$ . Indeed, simply set  $Q_h := Q - \lim \Phi^n(Q)$  and  $Q_p := Q - Q_h$ .

Our next goal is to describe all the pure superharmonic operators for a given completely positive map,  $\Phi$ , say. We will assume that we are given some  $W^*$ -correspondence  $E$  over  $M$ , a normal representation  $\sigma : M \rightarrow B(H)$  and an element  $\eta \in E^\sigma$  so that  $\Phi$  is realized as  $\Phi_\eta$  acting on  $\sigma(M)'$  through the formula

$$\Phi_\eta(a) := \eta^*(I_E \otimes a)\eta, \quad a \in \sigma(M)', \quad (18)$$

as in Theorem 4.1. The data  $(E, \eta, \sigma)$  need not be the data constructed in that result; it can be quite arbitrary. However, since we are not assuming that  $\Phi_\eta$  and  $\eta$  have norm at most one, some additional preparation is necessary. Since  $\eta^*$  is a bounded linear map from  $E \otimes_\sigma H$  to  $H$  that satisfies the equation  $\eta^* \sigma^E \circ \varphi = \sigma \eta^*$ , [18, Lemma 3.5] implies that if we define  $\widehat{\eta}^*$  by the formula

$$\widehat{\eta}^*(\xi)h = \eta^*(\xi \otimes h), \quad \xi \otimes h \in E \otimes_\sigma H,$$

then  $\widehat{\eta}^*$  is a completely bounded bimodule map with  $cb\text{-norm } \|\eta^*\| = \|\eta\|$ . We shall refer to the pair  $(\widehat{\eta}^*, \sigma)$  as a *completely bounded covariant representation* of  $(E, M)$ . Although  $\widehat{\eta}^*$  need not extend to a completely bounded representation of  $\mathcal{T}_+(E)$ , as would be the case if  $\|\eta\| \leq 1$ , we still can promote  $\widehat{\eta}^*$  to a map  $\widehat{\eta}_n^*$  on each of the tensor powers of  $E$ ,  $E^{\otimes n}$ , via the formula

$$\widehat{\eta}_n^*(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) := \widehat{\eta}^*(\xi_1)\widehat{\eta}^*(\xi_2) \cdots \widehat{\eta}^*(\xi_n).$$

When this is done, the map  $\widehat{\eta}_n^*$  is a bimodule map from  $E^{\otimes n}$  to  $B(H)$  whose associated linear map  $\widetilde{\eta}_n^*$  from  $E^{\otimes n} \otimes_\sigma H$  to  $H$  is given by formula

$$\begin{aligned}\widetilde{\eta}_n^*(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \otimes h) &= \eta^*(\xi_1 \otimes \eta^*(\xi_2 \otimes (\cdots \otimes \eta^*(\xi_n \otimes h) \cdots)) \\ &= \eta^*(I_E \otimes \eta^*)(I_{E^{\otimes 2}} \otimes \eta^*) \cdots (I_{E^{\otimes (n-1)}} \otimes \eta^*)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \otimes h),\end{aligned}$$

i.e.,  $\widetilde{\eta}_n^* = \eta^*(I_E \otimes \eta^*)(I_{E^{\otimes 2}} \otimes \eta^*) \cdots (I_{E^{\otimes (n-1)}} \otimes \eta^*)$ . To lighten the notation, we drop the “hat” and “tilde”, and simply write  $\eta_n^* = \eta^*(I_E \otimes \eta^*)(I_{E^{\otimes 2}} \otimes \eta^*) \cdots (I_{E^{\otimes (n-1)}} \otimes \eta^*)$ . This is entirely consistent with what we used for  $\widetilde{T}_n$  in paragraph 2.8.3. Further, we may then also define  $\eta_n := (\eta_n^*)^*$ , which yields

$$\eta_n = (I_{E^{\otimes (n-1)}} \otimes \eta)(I_{E^{\otimes (n-2)}} \otimes \eta) \cdots (I_E \otimes \eta)\eta$$

as expected. We let  $\eta_0$  be the map from  $H$  to  $M \otimes_\sigma H$  that identifies  $H$  with  $M \otimes_\sigma H$  in the customary fashion. With this notation, we find that

$$\Phi_\eta^n(a) = \eta_n^*(I_{E^{\otimes n}} \otimes a)\eta_n \quad (19)$$

for all  $a \in \sigma(M)'$  and all  $n \geq 0$ .

**Theorem 4.6** *Let  $(\widehat{\eta}^*, \sigma)$  be a completely bounded covariant representation of  $(E, M)$  on the Hilbert space  $H$  and let  $\Phi_\eta$  be the completely positive map on  $\sigma(M)'$  defined by (18). An operator  $Q$  in  $\sigma(M)'$  is a pure superharmonic operator for  $\Phi_\eta$  if and only if  $Q = CC^*$  for an operator  $C \in \mathcal{I}((S_0, \sigma_0), (\widehat{\eta}^*, \sigma))$ . In this event, if  $r = (Q - \Phi(Q))^{\frac{1}{2}}$ , then  $(I_{\mathcal{F}(E)} \otimes r)C(\eta)$  is a bounded linear operator defined on all of  $H$ , mapping  $H$  to  $\mathcal{F}(E) \otimes_\sigma H$ , and  $C^*$  may be written as  $C^* = (I_{\mathcal{F}(E)} \otimes v)(I_{\mathcal{F}(E)} \otimes r)C(\eta)$ , where  $v$  is any partial isometry in  $\mathcal{I}(\sigma, \pi)$  whose initial projection contains the range projection of  $r$ .*

**Proof.** Suppose  $Q \in \sigma(M)'$  has the form  $Q = CC^*$ ,  $C \in \mathcal{I}((S_0, \sigma_0), (\widehat{\eta}^*, \sigma))$ . Then by equation (19) we may write

$$\begin{aligned}\Phi_\eta^n(Q) &= \eta_n^*(I_{E^{\otimes n}} \otimes Q)\eta_n \\ &= \eta_n^*(I_{E^{\otimes n}} \otimes C)(I_{E^{\otimes n}} \otimes C^*)\eta_n \\ &= C(\widetilde{S}_0)_n((\widetilde{S}_0)_n)^*C^* \\ &= CP_nC^*,\end{aligned}$$

where here we use  $P_n$  to denote the projection onto  $\sum_{k \geq n} E^{\otimes k} \otimes_\pi K_0$ . Since the  $P_n$  decrease strongly to zero, the operators  $\Phi_\eta^n(Q)$  decrease strongly to zero, as  $n \rightarrow \infty$ . Thus  $Q$  is pure superharmonic for  $\Phi_\eta$ .

For the converse, suppose  $Q \in \sigma(M)'$  is a given pure superharmonic operator for  $\Phi_\eta$  and write  $r^2 := Q - \Phi_\eta(Q)$ . The “purity” of  $Q$  guarantees that  $Q = \sum_{n \geq 0} \Phi_\eta^n(r^2)$ , where the series converges in the strong operator topology. Indeed, the  $n^{\text{th}}$  partial sum of the series is  $Q - \Phi_\eta^{n+1}(Q)$ . Let  $\mathcal{R}$  be the closure of the range of  $r$ . Since  $r \in \sigma(M)'$ ,  $\mathcal{R}$  reduces  $\sigma$  and so we get a new normal representation,  $\sigma_{\mathcal{R}}$ , of  $M$  by restricting  $\sigma(\cdot)$  to  $\mathcal{R}$ . Choose an isometry  $v$  from  $\mathcal{R}$  into  $K_0$  that is in  $\mathcal{I}(\sigma_{\mathcal{R}}, \pi)$ . (Such a choice is possible by the definition of  $\pi$ .) Define  $C^* : H \rightarrow \mathcal{F}(E) \otimes_\pi K_0$  by the formula

$$\begin{aligned} C^*x &:= (I_{\mathcal{F}(E)} \otimes v) \sum_{n \geq 0} (I_{E^{\otimes n}} \otimes r) \eta_n x \\ &= (I_{\mathcal{F}(E)} \otimes vr) C(\eta)x. \end{aligned}$$

A straightforward calculation shows that this series converges and that the sum defines a bounded operator  $C^*$  that satisfies the equation

$$\begin{aligned} CC^*x &= \sum_{n \geq 0} \eta_n^* (I_{E^{\otimes n}} \otimes r^2) \eta_n x \\ &= \sum_{n \geq 0} \Phi_\eta^n(r^2)x \\ &= Qx. \end{aligned}$$

It is also clear that  $C \in \mathcal{I}((S_0, \sigma_0), (\widehat{\eta}^*, \sigma))$ .  $\square$

Theorem 4.6 has its roots in work of Kato [11]. Indeed, he might have called the operator  $r$  a *smooth operator* with respect to  $\eta^*$  (See [11, p. 545].) The proof of the theorem that we presented is a minor modification of Douglas’s proof of Theorem 5 in [5]. Popescu proved Theorem 4.6 in the setting of free semigroup algebras as [20, Theorem 3.7] and developed a number of other important features of  $\Phi_\eta$  in that setting. Many of them extend to our context, but we will not pursue all of them here.

Our primary objective is to prove the following theorem that identifies  $\mathcal{V}_{ac}(T, \sigma)$  for a completely contractive representation  $(T, \sigma)$  of  $(E, M)$ .

**Theorem 4.7** *Let  $(T, \sigma)$  be a completely contractive representation of  $(E, M)$  on the Hilbert space  $H$ , let  $(V, \rho)$  be the minimal isometric dilation of  $(T, \sigma)$*

acting on a Hilbert space  $K$  containing  $H$ , and let  $P$  denote the projection of  $K$  onto  $H$ . Then  $K \ominus H$  is contained in  $\mathcal{V}_{ac}(V, \rho)$  and the following sets are equal.

- (1)  $\mathcal{V}_{ac}(T, \sigma)$ .
- (2)  $H \cap \mathcal{V}_{ac}(V, \rho)$ .
- (3)  $P\mathcal{V}_{ac}(V, \rho)$ .
- (4)  $\bigcup\{Ran(C) \mid C \in \mathcal{I}((S_0, \sigma_0), (T, \sigma))\}$ .
- (5)  $\overline{\text{span}}\{Ran(Q) \mid Q \text{ is a pure superharmonic operator for } \Phi_T\}$ .

In particular,  $(T, \sigma)$  is absolutely continuous if and only if  $(V, \rho)$  is absolutely continuous.

**Proof.** First, observe that the orthogonal complement of  $H$  in  $K$ ,  $H^\perp$ , is  $\mathcal{F}(E) \otimes_{\sigma_1} \mathcal{D}$ , where  $\mathcal{D}$  is the closure of the range of  $\Delta = (I_{E \otimes H} - \tilde{T}^* \tilde{T})^{\frac{1}{2}}$  and where  $\sigma_1$  is the restriction of  $\sigma^E \circ \varphi(\cdot)$  to  $\mathcal{D}$ . (See paragraph 2.12.) The restriction of  $(V, \rho)$  to  $H^\perp = \mathcal{F}(E) \otimes_{\sigma_1} \mathcal{D}$  is just the representation induced by  $\sigma_1$  and, therefore, is absolutely continuous. Thus  $K \ominus H \subseteq \mathcal{V}_{ac}(V, \rho)$ . To see the equality of the indicated subspaces, begin by noting that the coincidence of the two spaces  $\mathcal{V}_{ac}(T, \sigma)$  and  $H \cap \mathcal{V}_{ac}(V, \rho)$  is an immediate consequence of the fact that for every vector  $x \in H$  the two functionals  $\omega_x \circ (T \times \sigma)$  and  $\omega_x \circ (V \times \rho)$  are equal. This, in turn, is clear because for such an  $x$ ,  $Px = x$ , where  $P$  is the projection from  $K$  onto  $H$ . Consequently,  $\omega_x \circ (V \times \rho)(\cdot) = \langle V \times \rho(\cdot)x, x \rangle = \langle PV \times \rho(\cdot)Px, x \rangle = \langle T \times \sigma(\cdot)x, x \rangle = \omega_x \circ (T \times \sigma)(\cdot)$ .

Note in particular, by Theorem 3.7, the fact that  $\mathcal{V}_{ac}(T, \sigma) = H \cap \mathcal{V}_{ac}(V, \rho)$  shows that  $\mathcal{V}_{ac}(T, \sigma)$  is a closed subspace of  $H$ .

Clearly,  $H \cap \mathcal{V}_{ac}(V, \rho)$  is contained in  $P\mathcal{V}_{ac}(V, \rho)$ . On the other hand, if  $x = Py$ , with  $y \in \mathcal{V}_{ac}(V, \rho)$ , then by Proposition 3.5, there is an  $X \in \mathcal{I}((S_0, \sigma_0), (V, \rho))$  and a  $z \in \mathcal{F}(E) \otimes_{\pi} K_0$  such that  $y = Xz$ . Since  $H^\perp$  is invariant under  $V \times \rho$ , we see that  $(T \times \sigma)PX = P(V \times \rho)PX = P(V \times \rho)X = PX(S_0 \times \rho)$ . Thus  $x = PXz$  lies in  $\bigcup\{Ran(C) \mid C \in \mathcal{I}((S_0, \sigma_0), (T, \sigma))\}$ . On the other hand, the commutant lifting theorem, [18, Theorem 4.4], implies that every operator  $C \in \mathcal{I}((S_0, \sigma_0), (T, \sigma))$  has the form  $PX$  for an operator  $X \in \mathcal{I}((S_0, \sigma_0), (V, \rho))$ . Thus,  $\bigcup\{Ran(C) \mid C \in \mathcal{I}((S_0, \sigma_0), (T, \sigma))\} \subseteq P \bigvee\{Ran(X) \mid X \in \mathcal{I}((S_0, \sigma_0), (V, \rho))\} = P\mathcal{V}_{ac}(V, \rho)$ , where the last equation is justified by Theorem 3.7. Thus  $\bigcup\{Ran(C) \mid C \in \mathcal{I}((S_0, \sigma_0), (T, \sigma))\} =$



$P\mathcal{V}_{ac}(V, \rho)$ . To see that  $P\mathcal{V}_{ac}(V, \rho) = H \cap \mathcal{V}_{ac}(V, \rho)$ , note that we showed that  $\mathcal{V}_{ac}(V, \rho)$  contains  $H^\perp$  and so the projection onto  $\mathcal{V}_{ac}(V, \rho)$  commutes with  $P$ . Consequently,  $H \cap \mathcal{V}_{ac}(V, \rho) = P\mathcal{V}_{ac}(V, \rho)$ , and so the first four sets (1)–(4) are equal.

From Theorem 4.6, we know that if  $C \in \mathcal{I}((S_0, \sigma_0), (T, \sigma))$ , then  $CC^*$  is pure superharmonic for  $\Phi_T$ . Although the range of  $CC^*$  may be properly contained in the range of  $C$  it is dense in the range of  $C$ , and so we see that  $\bigcup \{Ran(C) \mid C \in \mathcal{I}((S_0, \sigma_0), (T, \sigma))\} \subseteq \overline{span} \{Ran(Q) \mid Q \text{ is a pure superharmonic operator for } \Phi_T\}$ . The opposite inclusion is an immediate consequence of the opposite implication in Theorem 4.6, which shows that every pure superharmonic  $Q$  for  $\Phi_T$  has the form  $Q = CC^*$  for a suitable  $C \in \mathcal{I}((S_0, \sigma_0), (T, \sigma))$ , and the fact, already proved, that  $\bigcup \{Ran(C) \mid C \in \mathcal{I}((S_0, \sigma_0), (T, \sigma))\}$  is the closed linear space  $\mathcal{V}_{ac}(T, \sigma)$ .  $\square$

**Corollary 4.8** *Suppose  $(T, \sigma)$  is a completely contractive covariant representation of  $(E, M)$  on  $H$ . Then:*

1.  $\mathcal{V}_{ac}(T, \sigma) = 0$  if and only if  $\mathcal{I}((S_0, \sigma_0), (T, \sigma)) = \{0\}$ .
2. If  $\|\tilde{T}\| < 1$ , then  $(T, \sigma)$  is absolutely continuous.

**Proof.** The first assertion is immediate from part (4) of Theorem 4.7. The second assertion is immediate from part (5) since when  $\|\tilde{T}\| < 1$ , the identity is a pure superharmonic for operator  $\Phi_T$  by equation 7 and the fact that  $\|\tilde{T}_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

**Remark 4.9** *With Corollary 4.8 in hand, it is easy to pick up on a point raised at the end of paragraph 2.3. Let  $(T, \sigma)$  be completely contractive covariant representation of  $(E, M)$  on a Hilbert space  $H$  and assume that  $\|\tilde{T}\| < 1$ . Then  $(T, \sigma)$  is absolutely continuous by Corollary 4.8, which means that  $\mathcal{I}((S_0, \sigma_0), (T, \sigma))$  is quite large. On the other hand, it is easy to see that if  $\|\tilde{T}\| < 1$ , then  $\mathcal{I}((T, \sigma), (S_0, \sigma_0)) = 0$ . Indeed, if  $C \in \mathcal{I}((T, \sigma), (S_0, \sigma_0))$ , then  $C\tilde{T} = \tilde{S}_0(I_E \otimes C)$ . From this it follows that  $C\tilde{T}_n = (\tilde{S}_0)_n(I_{E^{\otimes n}} \otimes C)$  for all  $n$ . Since each  $(\tilde{S}_0)_n$  is isometric, this equation implies that*

$$(\tilde{S}_0)_n^* C \tilde{T}_n = I_{E^{\otimes n}} \otimes C.$$

*We conclude that  $C = 0$ , since the left hand side of this equation goes to zero in norm, while the right hand side has norm  $\|C\|$  for all  $n$ .*

We note in passing that when Theorem 4.7 is specialized to the setting when  $M = E = \mathbb{C}$ , it yields an improvement of [5, Corollary 5.5] in the following sense: If  $W$  is a unitary operator on a Hilbert space  $H$ , then its absolutely continuous subspace is the closed span of the ranges of all the pure superharmonic operators with respect to the automorphism of  $B(H)$  induced by  $W$ ; it is also the union of the ranges of all the operators that intertwine the unilateral shift of infinite multiplicity and  $W$ .

Recall that if  $\mathcal{A}$  is an algebra of operators on a Hilbert space  $H$ , then a subspace  $\mathcal{M}$  of  $H$  is called hyperinvariant for  $\mathcal{A}$  if and only if  $\mathcal{M}$  is invariant under every operator in  $\mathcal{A}$  and every operator in the commutant of  $\mathcal{A}$ . One important feature of this notion is that when  $\mathcal{A}$  is generated by single normal operator  $T$ , say, then the hyperinvariant subspaces of  $\mathcal{A}$  are precisely the spectral subspaces of  $T$ . Thus in a sense, hyperinvariant subspaces for an algebra should be viewed as analogues of spectral subspaces for an operator. One needs to take this extended perspective with a grain of salt, however, since spectral subspaces need not be central, i.e., the projection  $P$  onto a hyperinvariant subspace need not lie in the center of  $\mathcal{A}$ . Nevertheless, knowing that a subspace is hyperinvariant for an algebra is useful information. Evidently, if  $(T, \sigma)$  is a completely contractive covariant representation of  $(E, M)$  then  $\mathcal{V}_{ac}(T, \sigma)$  is invariant under  $T \times \sigma(\mathcal{T}_+(E))$  by part (4) of Theorem 4.7. Indeed, if  $h \in \mathcal{V}_{ac}(T, \sigma)$ , then there is a vector  $x \in \mathcal{F}(E) \otimes_\pi K_0$  and an operator  $C \in \mathcal{I}((S_0, \sigma_0), (T, \sigma))$  such that  $h = Cx$ . Then for  $\xi \in E$  and  $a \in M$ ,  $T(\xi)h = T(\xi)Cx = CS_0(\xi)x$  and  $\sigma(a)h = \sigma(a)Cx = C\pi^{\mathcal{F}(E)}x$  are in  $\mathcal{V}_{ac}(T, \sigma)$ . The next result, a consequence of Theorems 4.7 and 4.6, shows that  $\mathcal{V}_{ac}(T, \sigma)$  is hyperinvariant for  $T \times \sigma(\mathcal{T}_+(E))$ .

**Theorem 4.10** *For  $i = 1, 2$ , let  $(T_i, \sigma_i)$  be a completely contractive covariant representation of  $(E, M)$  on a Hilbert space  $H_i$  and suppose that  $R : H_1 \rightarrow H_2$  intertwines  $(T_1, \sigma_1)$  and  $(T_2, \sigma_2)$ . Then  $R\mathcal{V}_{ac}(T_1, \sigma_1) \subseteq \mathcal{V}_{ac}(T_2, \sigma_2)$ . In particular, the absolutely continuous subspace of a completely contractive covariant representation of  $(E, M)$  is hyperinvariant for its image.*

**Proof.** By Theorem 4.7,  $\mathcal{V}_{ac}(T_i, \sigma_i) = \bigcup \{ \text{Ran}(C) \mid C \in \mathcal{I}((S_0, \sigma_0), (T_i, \sigma_i)) \}$ ,  $i = 1, 2$ . If  $C \in \mathcal{I}((S_0, \sigma_0), (T_1, \sigma_1))$ , then  $RC(S_0 \times \sigma_0) = R(T_1 \times \sigma_1)C = (T_2 \times \sigma_2)RC$ , which shows that  $R\mathcal{I}((S_0, \sigma_0), (T_1, \sigma_1)) \subseteq \mathcal{I}((S_0, \sigma_0), (T_2, \sigma_2))$ . Since  $R(\text{Ran}(C)) = \text{Ran}(RC)$ , we conclude that  $R\mathcal{V}_{ac}(T_1, \sigma_1) \subseteq \mathcal{V}_{ac}(T_2, \sigma_2)$ .  $\square$

We come now to the main result of this section, which provides criteria for deciding when a completely contractive covariant representation of  $\mathcal{T}_+(E)$  extends to an ultra-weakly continuous representation of  $H^\infty(E)$ .

**Theorem 4.11** *Let  $(T, \sigma)$  be a completely contractive covariant representation of  $\mathcal{T}_+(E)$  on a Hilbert space  $H$  and let  $(V, \rho)$  be its minimal isometric dilation acting on a Hilbert space  $K$  containing  $H$ . Then the following assertions are equivalent.*

1.  $T \times \sigma$  extends to an ultra-weakly continuous, completely contractive representation of  $H^\infty(E)$ .
2.  $(T, \sigma)$  is absolutely continuous.
3.  $\overline{\text{span}}\{\text{Ran}(Q) \mid Q \text{ is a pure superharmonic operator for } \Phi_T\} = H$ .
4.  $V \times \rho$  extends to an ultra-weakly continuous, completely contractive representation of  $H^\infty(E)$ .
5.  $(V, \rho)$  is absolutely continuous.
6.  $H$  is contained in  $\mathcal{V}_{ac}(V, \rho)$ .

Of course, we could add a number of other conditions to this list. However, these are the principal ones and more important, none refers to the “external construct”  $(S_0, \sigma_0)$ . That is to say, all the conditions listed refer to *intrinsic* features of the representation  $(T, \sigma)$ .

**Proof.** Of course, much of the proof amounts to assembling pieces already proved. Thus, (2) and (3) are equivalent by virtue of Theorem 4.7. Likewise, (4) and (5) are equivalent by Theorem 3.10.

The equivalence of (5) and (6) follows from the observation that  $(V, \rho)$  is absolutely continuous if and only if  $H \subseteq \mathcal{V}_{ac}(V, \rho)$ . (This is because  $H^\perp \subseteq \mathcal{V}_{ac}(V, \rho)$ , as we noted in the proof of Theorem 4.7.) Thus (5) and (6) are equivalent. Conditions (2) and (6) are equivalent by virtue of the equation  $\mathcal{V}_{ac}(T, \sigma) = H \cap \mathcal{V}_{ac}(V, \rho)$  proved in Theorem 4.7. Thus conditions (2) – (6) are equivalent.

But (1) certainly implies (2). On the other hand, if (2) holds, so does (4). If  $\widetilde{V \times \rho}$  denotes the ultra-weakly continuous extension of  $V \times \rho$  to  $H^\infty(E)$ , then it is clear that  $P(\widetilde{V \times \rho})|_H$  is an ultra-weakly continuous extension of  $T \times \sigma$  to  $H^\infty(E)$ .  $\square$

## 5 Further Corollaries and Examples

### 5.1 Invariant States

In this subsection we collect a number of results that show how the notion of absolute continuity relates to the notion of an invariant state for a completely positive map.

**Definition 5.1** *Let  $\Phi$  be a normal completely positive map on a  $W^*$ -algebra  $M$ . We let  $P_{ac} = P_{ac}(\Phi)$  denote the smallest projection in  $M$  that dominates the range projection of each pure superharmonic element of  $\Phi$  and we call  $P_{ac}$  the absolutely continuous projection for  $\Phi$ .*

If  $M$  is represented faithfully on a Hilbert space  $H$  by a normal representation  $\rho$ , and if  $\Phi$  is contractive, then the range of  $\rho(P_{ac})$  is the absolutely continuous subspace  $\mathcal{V}_{ac}(T, \sigma)$  for the identity representation  $(T, \sigma)$  of  $(E, \rho(M)')$ , where  $E$  is the Arveson-Stinespring correspondence determined by  $\Phi$  (and  $\rho$ ), by Theorem 4.1 and Theorem 4.7. So the terminology is consistent with the developments in Sections 4 and 5, and it makes sense without having to assume  $\Phi$  is contractive.

**Definition 5.2** *If  $\Phi$  be a normal completely positive map on a  $W^*$ -algebra  $M$ , then a normal state  $\omega$  on  $N$  is called periodic of period  $k$  with respect to  $\Phi$ , if  $k$  is the least positive integer such that  $\omega \circ \Phi^k = \omega$ . We denote the collection of all normal period states of period  $k$  for  $\Phi$  by  $\mathcal{P}_k$  or by  $\mathcal{P}_k(\Phi, M)$ .*

Recall that if  $\omega$  is a normal state on a  $W^*$ -algebra  $N$  then there is a largest projection  $e \in N$  such that  $\omega(e) = 0$ . The projection  $e^\perp := 1 - e$  is called the *support projection* for  $\omega$ , which we shall denote by  $\text{supp}(\omega)$ . As is customary, we often identify a projection with its range and we shall think of  $\text{supp}(\omega)$  as a subspace of whatever Hilbert space on which  $N$  may be found to be acting. Our aim now is to prove

**Theorem 5.3** *Let  $\Phi$  be a normal completely positive map on the  $W^*$ -algebra  $M$ . If  $\omega$  is a normal state on  $M$  that is periodic for  $\Phi$ , then its support projection is orthogonal to  $P_{ac}(\Phi)$ .*

The proof is based on the following lemma.

**Lemma 5.4** *For each  $k \geq 2$ ,  $P_{ac}(\Phi^k) = P_{ac}(\Phi)$ .*

**Proof.** It is clear that  $P_{ac}(\Phi) \leq P_{ac}(\Phi^k)$  since every superharmonic operator for  $\Phi$  is superharmonic for  $\Phi^k$ . The problem we face with trying to prove the reverse inclusion is that in general a pure superharmonic operator for  $\Phi^k$  is not evidently a pure superharmonic operator for  $\Phi$ . So what we prove is that if  $Q$  is a pure superharmonic operator for  $\Phi^k$ , then there is a pure superharmonic operator  $R$  for  $\Phi$  such that  $Q \leq R$ . This will show that  $P_{ac}(\Phi^k) \subseteq P_{ac}(\Phi)$ . Our choice for  $R$  is  $Q + \Phi(Q) + \cdots + \Phi^{k-1}(Q)$ . Evidently, this operator dominates  $Q$ . So it suffices to show that it is a pure superharmonic operator for  $\Phi$ . Since  $\Phi(Q + \Phi(Q) + \cdots + \Phi^{k-1}(Q)) = \Phi(Q) + \Phi^2(Q) + \cdots + \Phi^k(Q)$  and  $\Phi^k(Q) \leq Q$  by hypothesis, we see that

$$\begin{aligned} \Phi(Q + \Phi(Q) + \cdots + \Phi^{k-1}(Q)) &= \Phi(Q) + \Phi(Q) + \cdots + \Phi^k(Q) \\ &\leq \Phi(Q) + \Phi(Q) + \cdots + \Phi^{k-1}(Q) + Q \\ &= Q + \Phi(Q) + \cdots + \Phi^{k-1}(Q). \end{aligned}$$

Thus  $Q + \Phi(Q) + \cdots + \Phi^{k-1}(Q)$  is superharmonic for  $\Phi$ . This means that the sequence  $\{\Phi^n(Q + \Phi(Q) + \cdots + \Phi^{k-1}(Q))\}_{n \geq 0}$  is decreasing. Thus to show it tends strongly to zero, it suffices to show that a subsequence tends to zero strongly. But the sequence  $\{\Phi^{nk}(Q + \Phi(Q) + \cdots + \Phi^{k-1}(Q))\}_{n \geq 0}$  has this property since

$$\Phi^{nk}(Q + \Phi(Q) + \cdots + \Phi^{k-1}(Q)) = \Phi^{nk}(Q) + \Phi(\Phi^{nk}(Q)) + \cdots + \Phi^{k-1}(\Phi^{nk}(Q))$$

and each term on the right hand side tends to zero monotonically as  $n \rightarrow \infty$ .  $\square$

**Proof of Theorem 5.3.** Since  $P_{ac}(\Phi^k) = P_{ac}(\Phi)$  by Lemma 5.4, it suffices to show that if  $\omega$  is an invariant state for  $\Phi$  then  $\text{supp}(\omega) \perp P_{ac}(\Phi)$ . But if  $Q$  is a pure superharmonic operator for  $\Phi$ , then the equation  $\omega \circ \Phi = \omega$  implies that  $\omega(Q) = 0$ . Thus the range projection of  $Q$  is orthogonal to the support of  $\omega$ , and so  $\text{supp}(\omega) \perp P_{ac}(\Phi)$ .  $\square$

Recall from [15, Page 404 ff.] that if  $(T, \sigma)$  is an ultra-weakly continuous completely contractive representation of  $(E, M)$  acting on the Hilbert space  $H$ , then there is a largest subspace  $H_1$  of  $H$  that is invariant under  $(T \times \sigma)(\mathcal{T}_+(E))^*$  on which  $\widetilde{T}^*$  acts isometrically. It is given by the equation

$$H_1 = \{h \in H \mid \|\widetilde{T}_n^* h\| = \|h\|, \text{ for all } n \geq 0\}.$$

The representations  $(T, \sigma)$  and  $T \times \sigma$  are called *completely non coisometric* (abbreviated c.n.c.) in case  $H_1 = \{0\}$ . We record for reference the following theorem that is proved as part of Theorem 7.3 in [15].

**Theorem 5.5** *If  $(T, \sigma)$  is a completely non coisometric, ultra-weakly continuous, completely contractive representation of  $(E, M)$  on a Hilbert space  $H$ , then  $T \times \sigma$  extends to an ultra-weakly continuous, completely contractive representation of  $H^\infty(E)$ , and so  $(T, \sigma)$  is absolutely continuous, by Theorem 4.11.*

Of course, if  $\|\tilde{T}\| < 1$ , then  $(T, \sigma)$  is completely non coisometric. Thus, Theorem 5.5 improves upon Corollary 4.8.

**Theorem 5.6** *Let  $(T, \sigma)$  be an ultra-weakly continuous, completely contractive covariant representation of  $(E, M)$  on a Hilbert space  $H$ . If  $\sigma(M)'$  is finite dimensional, then  $T \times \sigma$  extends to an ultra-weakly continuous representation of  $H^\infty(E)$  on  $H$  if and only if  $(T, \sigma)$  is completely non coisometric.*

**Proof.** We know that  $T \times \sigma$  extends if  $(T, \sigma)$  is completely non coisometric regardless of the dimension of  $\sigma(M)'$ . So we attend to the reverse implication. If  $(T, \sigma)$  has a nonzero coisometric part  $H_1$  then we can compress  $(T, \sigma)$  to  $H_1$  to get a new representation  $(T_1, \sigma_1)$  that is fully coisometric, i.e.,  $\tilde{T}_1 \tilde{T}_1^* = I_{H_1}$ . Of course  $\sigma_1(M)'$  will be finite dimensional, too. So, it suffices to assume from the outset that  $(T, \sigma)$  is fully coisometric, which is tantamount to assuming  $\Phi_T$  is unital. But a unital completely positive map on a finite dimensional  $W^*$ -algebra admits an invariant normal state. The support of such a state, as we have seen in Theorem 5.3, must be orthogonal to  $\mathcal{V}_{ac}(T, \sigma)$  and so  $(T, \sigma)$  cannot be absolutely continuous. By Theorem 4.11, we conclude that  $T \times \sigma$  does not admit an extension to an ultra-weakly continuous representation of  $H^\infty(E)$ .  $\square$

**Corollary 5.7** *If  $(T_1, T_2, \dots, T_d)$  is a row contraction where the  $T_i$  act on a finite dimensional Hilbert space, then the map which takes the  $i^{\text{th}}$  generator  $S_i$  of  $H^\infty(\mathbb{C}^d)$  to  $T_i$  extends to an ultra-weakly continuous representation of  $H^\infty(\mathbb{C}^d)$  if and only if  $(T_1, T_2, \dots, T_d)$  is completely non coisometric.*

**Example 5.8** *As an extremely simple, yet somewhat surprising concrete example, consider the case when  $d = 2$  and  $T_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and  $T_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  are acting on  $H = \mathbb{C}^2$ . Then  $(T_1, T_2)$  is a row coisometry and so is not absolutely continuous. In fact, since  $\Phi_T$  preserves the trace, as is easy to see, the absolutely continuous subspace of  $\mathbb{C}^2$  reduces to zero. Thus these matrices do not come from an ultra-weakly continuous representation of  $H^\infty(\mathbb{C}^2)$  even though  $T_1$  and  $T_2$  are both nilpotent.*

**Example 5.9** Suppose the  $W^*$ -algebra  $M$  is given and that  $E$  comes from a unital endomorphism  $\alpha$ , i.e., suppose  $E = {}_\alpha M$ , which is  $M$  as a (right) Hilbert module over  $M$ , with the left action of  $M$  given by  $\alpha$ . If  $(T, \sigma)$  is an ultra-weakly continuous, completely contractive covariant representation of  $(E, M)$  on a Hilbert space  $H$ , then the map  $x \otimes h \rightarrow \sigma(x)h$  extends to an isomorphism from  $E \otimes_\sigma H$  to  $H$  that allows us to view  $\tilde{T}$  as an operator  $T_0$  on  $H$  that has the property

$$T_0 \sigma(\alpha(a)) = \sigma(a) T_0$$

for all  $a \in M$ . The map  $\Phi_T$ , then, is given by the formula  $\Phi_T(x) = T_0 x T_0^*$ . If  $T_0$  is a unitary operator, then [5, Corollary 5.5] tells us that  $\mathcal{V}_{ac}(T, \sigma)$  is contained in the absolutely continuous subspace for  $T_0$ . It is, however, quite possible for the two spaces to be distinct. Indeed, let  $M$  be  $L^\infty(\mathbb{T})$  with  $\sigma$  the multiplication representation on  $L^2(\mathbb{T})$ . Let  $\alpha$  be implemented by an irrational rotation on  $\mathbb{T}$ , say  $z \rightarrow e^{i\theta} z$ . Finally, let  $T_0$  be the unitary operator on  $L^2(\mathbb{T})$  defined by the equation

$$T_0 \xi(z) = z \xi(e^{i\theta} z).$$

Then  $\sigma(M)' = L^\infty(\mathbb{T})$  and because  $\Phi_T(\varphi)(z) = \varphi(e^{i\theta} z)$ , as may be easily calculated, we see that  $\mathcal{V}_{ac}(T, \sigma) = \{0\}$  because integration against Lebesgue measure gives an invariant faithful normal state on  $M$ . On the other hand, it is easy to see that  $T_0$  is unitarily equivalent to the bilateral shift of multiplicity one. Indeed,  $T_0$  leaves  $H^2(\mathbb{T})$  invariant and  $H^2(\mathbb{T}) \ominus T_0 H^2(\mathbb{T}) = H^2(\mathbb{T}) \ominus z H^2(\mathbb{T})$ , where we have identified  $z$  with multiplication by  $z$ . It is an easy matter to check that  $\mathcal{D} := H^2(\mathbb{T}) \ominus T_0 H^2(\mathbb{T})$  is a complete wandering subspace for  $T_0$ , i.e.,  $L^2(\mathbb{T}) = \sum_{k \in \mathbb{Z}}^\oplus T_0^k \mathcal{D}$ . Thus, we see that  $T_0$  is an absolutely continuous unitary operator.

## 5.2 Markov Chains

Our next example connects the theory we have been developing with the theory of Markov chains. Recall from Example 4.3 that a sub-Markov matrix is an  $n \times n$  matrix  $A$  with non-negative entries  $a_{ij}$  with the property that the row-sums  $\sum_j a_{ij} \leq 1$ . We think of  $A$  as defining a completely positive map  $\Phi$  on the  $W^*$ -algebra of all diagonal  $n \times n$  matrices,  $D_n$ . If  $d = \text{diag}\{d_1, d_2, \dots, d_n\}$ , then  $\Phi(d) := \text{diag}\{\sum_j a_{1j} d_j, \sum_j a_{2j} d_j, \dots, \sum_j a_{nj} d_j\}$ . We want to describe the absolutely continuous projection  $\mathcal{P}_{ac}(\Phi)$ .

Recall that the norm of  $A$  as an operator on  $\ell^\infty(\{1, 2, \dots, n\})$  is at most 1 and if the norm of  $A$  is 1 then 1 is an eigenvalue for  $A$ . In this event, we can find an element  $\underline{z} = (z_1, z_2, \dots, z_n)$  such that  $z_i \geq 0$  for all  $i$  and such that  $A\underline{z} = \underline{z}$ . This is a consequence of the Perron-Frobenius theory (see [9, pp. 64,65].) It applies as well to the transpose of  $A$ . We will call such an eigenvector a *non-negative left eigenvector* for  $A$ . We will call the *support of  $\underline{z}$*  the set of  $i \in \{1, 2, \dots, n\}$  such that  $z_i \neq 0$ , and we will say that  $\underline{z}$  *has full support* if  $z_i \neq 0$  for all  $i$ . The analysis in Section III.4 of [9] shows that after conjugating  $A$  by a permutation matrix,  $A$  has the form described in the following lemma:

**Lemma 5.10** *If  $A$  is a sub-Markov matrix with spectral radius 1, then there is a permutation matrix  $S$  so that  $SAS^{-1}$  has the (block) lower triangular form*

$$\begin{bmatrix} A_1 & & & & \\ 0 & A_2 & & & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & A_{k-1k-1} & \\ & U & & & A_{kk} \end{bmatrix}, \quad (20)$$

where:

1. For  $i = 1, 2, \dots, k-1$ , each  $A_{ii}$  has 1 as an eigenvalue, and the corresponding left eigenvector is nonnegative and has full support.
2.  $A_{kk}$  has spectral radius less than 1.

**Definition 5.11** *We call the matrix (20) the canonical form of the sub-Markov matrix  $A$ . The set of indices  $E_i$  such that the matrix entries of  $A_{ii}$  are indexed by  $E_i \times E_i$  will be called the support of  $A_{ii}$  as well as the support of the left Perron-Frobenius eigenvector  $z^{(i)}$  for  $A_{ii}$ , corresponding to 1,  $i = 1, 2, \dots, k-1$ .*

**Lemma 5.12** *Let  $\Phi$  be a normal completely positive map on a  $W^*$ -algebra  $N$  and let  $p$  be a central projection of  $N$  that is invariant under  $\Phi$  in the sense that  $\Phi(p) \leq p$ . Then  $pP_{ac}(\Phi) = P_{ac}(\Phi|_{pNp})$ .*

**Proof.** Suppose  $Q$  is a pure superharmonic element of  $N$  for  $\Phi$ . Then the sequence  $\{\Phi^n(Q)\}_{n \in \mathbb{N}}$  decreases to zero. So  $\Phi(pQp) = p\Phi(pQp)p \leq$



$p\Phi(pQp + p^\perp Q p^\perp)p = p\Phi(Q)p \leq pQp$ . Therefore, since  $\Phi(Q)$  is also pure superharmonic for  $\Phi$ ,  $\Phi^2(pQp) = \Phi(\Phi(pQp)) = \Phi(p\Phi(pQp)p) \leq \Phi(p\Phi(Q)p) \leq p\Phi(\Phi(Q))p \leq \Phi^2(Q)$ . Continuing in this fashion, we see that  $\Phi^n(pQp) \leq \Phi^n(Q)$  for all  $n$ , and so the range projection of  $pQp$  is less than or equal to  $P_{ac}(\Phi|_{pNp})$ . That is,  $pP_{ac}(\Phi) \leq P_{ac}(\Phi|_{pNp})$ . The reverse inequality is clear, since if  $pQp$  is pure superharmonic for  $\Phi|_{pNp}$ , then it certainly is pure superharmonic for  $\Phi$ .  $\square$

**Theorem 5.13** *Let  $\Phi$  be the completely positive map on  $\ell^\infty(\{1, 2, \dots, n\})$  induced by a sub-Markov matrix  $A$  and assume  $A$  is written in its canonical form (20). Then  $P_{ac}(\Phi)$  is the support projection of  $A_{kk}$ .*

**Proof.** If  $E$  is the union of the support projections for  $A_i$ ,  $i = 1, 2, \dots, k-1$ , then the projection  $1_E$  in  $\ell^\infty(\{1, 2, \dots, n\})$  is coinvariant for  $A$  and the sum of the vectors  $z := \sum_{i=1}^{k-1} z^{(i)}$  determines an invariant state  $\omega_z$  for  $\Phi$  via formula  $\omega_z(\underline{d}) = \sum_{i=1}^{k-1} z^{(i)} \cdot \underline{d}_i$ , where  $\underline{d}_i$  is the restriction of  $\underline{d}$  to the support of  $z^{(i)}$ ,  $E_i$ , and  $z^{(i)} \cdot \underline{d}_i$  denotes the dot product of the two tuples. The state  $\omega_z$  is faithful on  $1_E \ell^\infty(\{1, 2, \dots, n\})$ , and so  $P_{ac}(\Phi) \leq 1_{E_k} = 1_{E^c}$ , by Theorem 5.3. On the other hand, since  $1_{E_k} \ell^\infty(\{1, 2, \dots, n\})$  is invariant for  $\Phi$ , and since the spectral radius of  $A_k$ , which is the matrix of the restriction of  $\Phi$  to  $1_{E_k} \ell^\infty(\{1, 2, \dots, n\})$ , is less than 1, it is clear that  $\Phi^n(1_{E_k}) \rightarrow 0$  in norm. This implies that  $1_{E_k}$  is pure superharmonic for  $\Phi$  and, therefore, that  $1_{E_k} \leq P_{ac}(\Phi)$ .  $\square$

### 5.3 Similarity of representations

Many of the results in this subsection are analogues of theorems in Popescu's paper [20] and many of his proofs work here, as well. We focus on those features and proofs that take advantage of our perspective that focuses on the connection between completely positive maps and intertwiners. So a number of our arguments are different from the ones given in [20]. Suppose  $\sigma : M \rightarrow B(H)$  is a normal representation of  $M$  on the Hilbert space  $H$  and suppose  $\eta \in E^\sigma$ . Then we may form the completely bounded bimodule map  $\widehat{\eta}^* : E \rightarrow B(H)$  discussed in the paragraph before Theorem 4.6. Let  $\mathcal{T}_{+0}(E)$  be the linear span of  $\varphi_\infty(M)$  and the operators  $\{T_\xi \mid \xi \in E^{\otimes n}, n \geq 1\}$ . Then  $\mathcal{T}_{+0}(E)$  is the algebra generated by  $\varphi_\infty(M)$  and  $\{T_\xi \mid \xi \in E\}$ , and given  $\sigma$  and  $\widehat{\eta}^*$ , we can define a representation of  $\mathcal{T}_{+0}(E)$  on  $H$ , denoted  $\widehat{\eta}^* \times \sigma$  by the formulae

$$\widehat{\eta}^* \times \sigma(\varphi_\infty(a)) = \sigma(a), \quad a \in M,$$

and

$$\widehat{\eta}^* \times \sigma(T_{\xi_1} T_{\xi_2} \cdots T_{\xi_n}) = \widehat{\eta}^*(\xi_1) \widehat{\eta}^*(\xi_2) \cdots \widehat{\eta}^*(\xi_n), \quad \xi_i \in E, i = 1, 2, \dots, n.$$

We are interested in understanding when  $\widehat{\eta}^* \times \sigma$  extends to a completely bounded representation of  $\mathcal{T}_+(E)$  in  $B(H)$ . Thanks to the famous theorem of Paulsen [19], this will happen if and only if  $\widehat{\eta}^* \times \sigma$  is similar to a completely contractive representation of  $\mathcal{T}_+(E)$ . By [18, Theorem 3.10] (see also paragraph (2.8)), this will happen if and only if  $\widehat{\eta}^* \times \sigma$  is similar to  $\widehat{\zeta}^* \times \sigma_1$ , for a  $\zeta \in \overline{\mathbb{D}(E^{\sigma_1})}^*$ .

Thus, we are led to investigating the similarity properties of the completely bounded maps  $\widehat{\eta}^* \times \sigma$ . For this purpose, observe that if  $\widehat{\eta}_1^* \times \sigma_1$  is similar to  $\widehat{\eta}_2^* \times \sigma_1$  then there is an invertible operator  $S$  on  $H$  such that  $\widehat{\eta}_1^* \times \sigma_1(a)S = S\widehat{\eta}_2^* \times \sigma_2(a)$  for all  $a \in \mathcal{T}_+(E)$ . In particular, when  $a \in \varphi_\infty(M)$ ,  $\sigma_1(a)S = S\sigma_2(a)$ , i.e.,  $\sigma_1$  and  $\sigma_2$  are similar. Since they are  $*$ -maps,  $\sigma_1$  and  $\sigma_2$  must be unitarily equivalent. But observe that if  $U$  is a Hilbert space isomorphism from the space of  $\sigma_1$ ,  $H_1$ , to the space of  $\sigma_2$ , then the map  $\eta \rightarrow (I_E \otimes U)\eta U^{-1}$  is a complete isometric isomorphism between  $E^{\sigma_1}$  and  $E^{\sigma_2}$  that is also a homeomorphism for the ultra-weak topologies. Since we will be interested here in the norm properties of  $\eta$ 's and related constructs the difference between the duals of unitarily equivalent representations may safely be ignored. Thus, in particular, when given a similarity  $S$  between  $\widehat{\eta}_1^* \times \sigma_1$  and  $\widehat{\eta}_2^* \times \sigma_2$  we may identify  $\sigma_1$  and  $\sigma_2$  with one  $\sigma$  and assume that  $S$  lies in  $\sigma(M)'$ . In this case,  $\eta_1^*(I \otimes S) = S\eta_2^*$ , i.e.  $S^{-1}\eta_1^*(I \otimes S) = \eta_2^*$ . Conversely, any invertible  $S$  in the commutant of  $\sigma(M)$  that satisfies the equation  $S^{-1}\eta_1^*(I \otimes S) = \eta_2^*$  implements a similarity between  $\widehat{\eta}_1^* \times \sigma(\cdot)$  and  $\widehat{\eta}_2^* \times \sigma(\cdot)$ .

**Definition 5.14** *We introduce several terms we will use in the sequel.*

1. If  $\eta$  and  $\zeta$  are two elements of  $E^\sigma$ , then we shall say they are similar if there is an invertible operator  $S$  in  $\sigma(M)'$  such that  $S^{-1} \cdot \eta \cdot S = (I_E \otimes S)^{-1} \eta S = \zeta$ .
2. Let  $\Phi_1$  and  $\Phi_2$  be two completely positive maps defined on a  $W^*$ -algebra  $N$ . We say they are similar if and only if there is an invertible operator  $R \in N$  such that

$$\psi_R^{-1} \circ \Phi_1 \circ \psi_R = \Phi_2,$$

where  $\psi_R$  is the complete positive map on  $M$  defined by the formula  $\psi_R(a) = RaR^*$ .

3. We say that an  $\eta \in E^\sigma$  belongs to class  $C_0$  in case the identity is a pure superharmonic operator for  $\Phi_\eta$ .

**Remark 5.15** Several points may be helpful.

1. We have taken the definition of similarity for completely positive maps from Popescu [20].
2. It is easy to see that if elements  $\eta, \zeta \in E^\sigma$  are similar, then  $\Phi_\eta$  and  $\Phi_\zeta$  are similar. Indeed, if  $R$  is an invertible element of  $\sigma(M)'$ , then

$$\begin{aligned}\psi_R^{-1} \circ \Phi_\eta \circ \psi_R(a) &= R^{-1}(\eta^*(I_E \otimes RaR^*)\eta)R^{*-1} \\ &= R^{-1}\eta^*(I_E \otimes R)(I_E \otimes a)(I_E \otimes R^*)\eta R^{*-1} \\ &= \Phi_\zeta,\end{aligned}$$

where  $\zeta = (I_E \otimes R^*)\eta R^{*-1}$ . The converse assertion is not true owing to the nonuniqueness of representing a completely positive map  $\Phi$  in the form  $\Phi_\eta$ .

3. In fact, it is convenient to use bimodule notation:  $\eta$  and  $\zeta$  are similar if and only if there is an invertible  $r$  such that  $r \cdot \eta \cdot r^{-1} = \zeta$ . Observe that in this case,  $r \cdot \eta_n \cdot r^{-1} = \zeta_n$  for all  $n$ .
4. The notion of a  $C_0$  element of  $E^\sigma$  is borrowed from [15, Definition 7.14]. There the norm of the element was assumed to be at most 1. It turns out that when  $M = \mathbb{C} = E$ , so that  $\eta$  is really an operator on Hilbert space, then the terminology we have adopted agrees with that of Sz.-Nagy and Foiaş in [26].
5. In the terminology of [20], a completely positive map  $\Phi$  is called pure if and only if  $I$  is a pure superharmonic operator for  $\Phi$ .

Our first result gives a necessary and sufficient condition for  $\widehat{\eta^*} \times \sigma$  to extend to a completely bounded representation on  $\mathcal{T}_+(E)$ . It was inspired by [20, Theorem 5.13]. However, our proof is somewhat different.

**Theorem 5.16** *Let  $\sigma : M \rightarrow B(H)$  be a normal representation and let  $\eta \in E^\sigma$ . Then the following conditions are equivalent.*

1. The representation  $\widehat{\eta^*} \times \sigma$  extends to a completely bounded representation of  $\mathcal{T}_+(E)$ .

2.  $\eta$  is similar to a  $\zeta \in \overline{\mathbb{D}(E^\sigma)}$ .

3.  $\Phi_\eta$  admits an invertible superharmonic operator.

**Proof.** If  $\widehat{\eta}^* \times \sigma$  extends to a completely bounded representation of  $\mathcal{T}_+(E)$ , then  $\widehat{\eta}^* \times \sigma$  must be similar to a completely contractive representation  $\rho$  of  $\mathcal{T}_+(E)$  by Paulsen's famous theorem [19]. Since  $\rho$  must be of the form  $\widehat{\zeta}^* \times \sigma_1$  for some  $\zeta$  of norm at most 1 in  $E^{\sigma_1}$ ,  $\sigma$  and  $\sigma_1$  are similar, and therefore unitarily equivalent. As we noted above, we may identify  $\sigma$  with  $\sigma_1$  and conclude that  $\eta$  is similar to a point in  $\overline{\mathbb{D}(E^\sigma)^*}$ . Thus 1. implies 2. The converse is immediate, since as we noted above, a similarity between two points  $\eta_1$  and  $\eta_2$  in  $E^\sigma$  implements a similarity between  $\widehat{\eta}_1^* \times \sigma$  and  $\widehat{\eta}_2^* \times \sigma$ . Suppose  $\eta$  is similar to a  $\zeta$  in  $\overline{\mathbb{D}(E^\sigma)^*}$ , say  $r \cdot \eta \cdot r^{-1} = \zeta$  for some  $r \in \sigma(M)'$ . Then  $r \cdot \eta = \zeta \cdot r$  and so  $\Phi_\eta(r^*r) = \Phi_{r \cdot \eta}(I) = \Phi_{\zeta \cdot r}(I) = r^* \Phi_\zeta(I) r \leq r^* r$ , since  $\|\zeta\| \leq 1$ . Thus  $r^*r$  is superharmonic for  $\Phi_\eta$  and since  $r$  is invertible by assumption, 3. is proved. Finally, suppose  $\Phi_\eta$  admits an invertible superharmonic operator, say  $R$ . If  $r = R^{\frac{1}{2}}$  and if we set  $\zeta = r \cdot \eta \cdot r^{-1}$ , then  $\zeta \cdot r = r \cdot \eta$ ,  $\Phi_\zeta(I) = r^{-1} \Phi_{\zeta \cdot r}(I) r^{-1} = r^{-1} \Phi_{r \cdot \eta}(I) r^{-1} = r^{-1} \Phi_\eta(r^2) r^{-1} \leq r^{-1} r^2 r^{-1} = I$ , since  $r^2$  is superharmonic for  $\Phi_\eta$ . Thus  $\|\zeta\| \leq 1$ .  $\square$

The following theorem identifies when an  $\eta \in E^\sigma$  is similar to a  $C_0$  element in  $\overline{\mathbb{D}(E^\sigma)}$ . It is inspired by [20, Theorem 5.11]. Again, the proof is somewhat different.

**Theorem 5.17** *Let  $\sigma$  be a normal representation of  $M$  on a Hilbert space  $H$  and let  $\eta$  be an element of  $E^\sigma$ . Then the following assertions about  $\eta$  are equivalent.*

1.  $\eta$  is similar to a  $C_0$  element of  $\overline{\mathbb{D}(E^\sigma)}$ .

2. There is a positive element  $r \in \sigma(M)'$  and positive numbers  $a$  and  $b$  so that

$$aI_H \leq \sum_{n=0}^{\infty} \Phi_\eta^n(r) \leq bI_H. \quad (21)$$

3. There is an invertible pure superharmonic operator for  $\Phi_\eta$ .

**Proof.** The equivalence of 1. and 3. is an easy calculation of the sort that we performed above. If  $r \cdot \eta \cdot r^{-1} = \zeta$ , then  $r \cdot \eta_n = \zeta_n \cdot r$  for all  $n \geq 0$  and we further have

$$\Phi_\eta^n(r^*ar) = \Phi_{r \cdot \eta_n}(a) = \Phi_{\zeta_n \cdot r}(a) = r^* \Phi_{\zeta_n}(a) r = r^* \Phi_\zeta^n(a) r, \quad (22)$$

for all  $a \in \sigma(M)'$ . Now suppose that 1. holds, then with  $a = 1$ , we see that  $r^*r$  is an invertible superharmonic operator for  $\Phi_\eta$  because  $\Phi_\eta(r^*r) = r^*\Phi_\zeta(I)r \leq r^*r$ , since  $\|\zeta\| \leq 1$ . On the other hand, because  $\zeta$  is a  $C_0$  element of  $E^\sigma$ ,  $r^*r$  is a pure superharmonic operator:  $\Phi_\eta^n(r^*r) = r^*\Phi_\zeta^n(I)r \rightarrow 0$  weakly and, therefore, strongly. Thus, 3. is satisfied. The argument is essentially reversible: Suppose 3. holds and let  $a$  be a positive invertible superharmonic operator for  $\Phi_\eta$ . If we let  $r$  be the positive square root of  $a$ , then  $r$  is invertible and we may let  $\zeta = r \cdot \eta \cdot r^{-1}$ . Since  $a$  is superharmonic, we conclude that  $r^2 \geq \Phi_\eta(r^2) = r\Phi_\zeta(I)r$ , which shows that  $\Phi_\zeta(I) \leq I$ , because  $r$  is positive and invertible, and this implies that  $\zeta \in \overline{\mathbb{D}(E^\sigma)}$ . On the other hand, we conclude from these calculations and the assumption that  $a = r^2$  is a pure superharmonic operator for  $\Phi_\eta$ , that  $\Phi_\zeta^n(I) = r^{-1}\Phi_\eta^n(a)r \rightarrow 0$  weakly as  $n \rightarrow \infty$ . Thus  $\zeta$  is a  $C_0$  element of  $\overline{\mathbb{D}(E^\sigma)}$ , as was required.

Suppose assertion 2. is satisfied and let  $r$ ,  $a$  and  $b$  be as in equation (21). Then the series  $\sum_{n=0}^\infty \Phi_\eta^n(r)$  converges strongly to an operator  $R$  that is invertible in  $\sigma(M)'$ . Now  $\Phi_\eta(R) = \sum_{n=1}^\infty \Phi_\eta^n(r) = R - r \leq R$ . Thus  $R$  is superharmonic for  $\Phi_\eta$ . But also  $\Phi_\eta^n(R) = \sum_{k=n}^\infty \Phi_\eta^k(r)$  and this sequence of operators converges strongly to zero, since the series  $\sum_{n=0}^\infty \Phi_\eta^n(r)$  converges strongly. Thus condition 3. is satisfied.

Suppose condition 3. is satisfied, let  $R$  be an invertible pure superharmonic operator for  $\Phi_\eta$  and set  $r := R - \Phi_\eta(R)$ . Then  $r$  is positive semidefinite and  $R = \sum_{n=0}^\infty \Phi_\eta^n(r)$ . Since  $R$  is assumed invertible, the inequality (21) is satisfied for suitable  $a$  and  $b$ .  $\square$

The next theorem uses intertwiners to describe when an  $\eta \in E^\sigma$  is similar to a  $\zeta$  in the *open* unit disc  $\mathbb{D}(E^\sigma)^*$ . It is similar in spirit to [20, Theorem 5.9], but arguments use different technology. Recall that  $(S_0, \sigma_0)$  denotes the universal isometric induced representation. Also, we let  $P_0$  be the orthogonal projection of  $\mathcal{F}(E) \otimes_\pi K_0$  onto the zero<sup>th</sup> summand,  $M \otimes_\pi K_0 \simeq K_0$ .

**Theorem 5.18** *Let  $\sigma$  be a normal representation of  $M$  on a Hilbert space  $H$  and let  $\eta$  be a point in  $E^\sigma$ . Then  $\eta$  is similar to a point in the open unit disc  $\mathbb{D}(E^\sigma)$  if and only if there is a  $C \in \mathcal{I}((S_0, \sigma_0), (\eta^*, \sigma))$  such that  $CP_0C^*$  is invertible.*

**Proof.** Suppose there is an invertible  $r \in \sigma(M)'$  such that  $r \cdot \eta \cdot r^{-1} = \zeta$ , with  $\|\zeta\| < 1$ . Then  $r \cdot \eta_n = \zeta_n \cdot r$  for all  $n$  and we see that  $\Phi_\eta^n(r^*r) = r^*\Phi_\zeta^n(I)r \leq r^*r\|\zeta^*\zeta\|^n$ , which shows both that  $r^*r$  is a pure superharmonic

operator for  $\Phi_\eta$ . So by Theorem 4.6 there is a  $C \in \mathcal{I}((S_0, \sigma_0), (\eta^*, \sigma))$  such that  $r^*r = CC^*$ . But also,

$$\begin{aligned} CP_0C^* &= C(I - \widetilde{S_0}\widetilde{S_0^*})C^* = CC^* - \eta^*CC^*\eta \\ &= r^*r - \Phi_\eta(r^*r) = r^*r - r^*\Phi_\zeta(I)r \geq (1 - \|\zeta\|^2)r^*r. \end{aligned}$$

Since  $r$  is invertible, so is  $CP_0C^*$ .

Conversely, suppose that there is a  $C \in \mathcal{I}((S_0, \sigma_0), (\eta^*, \sigma))$  such that  $CP_0C^*$  is invertible and let  $b \in \mathbb{R}$  satisfy the inequality  $CP_0C^* \geq bI > 0$ . Also let  $t$  be a positive number less than  $\frac{b}{\|CC^*\|}$  ( $< 1$ ). Our objective is to show that if  $r = (CC^*)^{\frac{1}{2}}$  and if  $\zeta = r \cdot \eta \cdot r^{-1}$ , then  $\|\zeta\|^2 \leq 1 - t$ . Recall that  $CP_0C^* = CC^* - \Phi_\eta(CC^*)$ . By definition of  $t$ ,  $CC^* \leq \|CC^*\|I \leq \frac{b}{t}I$ . Therefore  $CC^* - bI \leq CC^* - tCC^*$ . But then  $(1 - t)CC^* - \Phi_\eta(CC^*) \geq [CC^* - bI] - \Phi_\eta(CC^*) > 0$ . Consequently,  $\Phi_\eta(CC^*) \leq (1 - t)CC^*$ . Now  $CC^* = r^2$  and  $\zeta$  is defined to be  $r \cdot \eta \cdot r^{-1}$ . We have  $\|\zeta\|^2 = \|\Phi_\zeta(I)\| = \|r^{-1}\Phi_\eta(r^2)r^{-1}\| \leq \|r^{-1}((1 - t)r^2)r^{-1}\| = 1 - t$ . Thus, 2. implies 1.  $\square$

Our final theorem in this vein has no analogue in [20], but it is in the spirit of that paper. The proof rests on the main results proved to this point.

**Theorem 5.19** *Let  $\sigma : M \rightarrow B(H)$  be a normal representation of  $M$  on the Hilbert space  $H$ , and let  $\eta \in E^\sigma$ . Then the following assertions are equivalent.*

1.  $\eta$  is similar to an absolutely continuous  $\zeta \in \overline{\mathbb{D}(E^\sigma)}$ .
2.  $\Phi_\eta$  admits an invertible superharmonic operator and

$$H = \bigcup \{ \text{Ran}(C) \mid C \in \mathcal{I}((S_0, \sigma_0), (\eta^*, \sigma)) \}.$$

3.  $\Phi_\eta$  admits an invertible superharmonic operator and

$$H = \bigvee \{ \text{Ran}(Q) \mid Q \in \sigma(M)', Q - \text{pure superharmonic for } \Phi_\eta \}.$$

**Proof.** Because of the hypotheses in 2. and 3. that  $\Phi_\eta$  admits an invertible superharmonic function, we know from Theorem 5.16 that  $\eta$  is similar to a contraction in each of the situations. The point of 2. is that if  $\eta^*$  is similar to a point  $\zeta^* \in \overline{\mathbb{D}(E^\sigma)}$  then there is an invertible  $r \in \sigma(M)'$  such

that  $r(\eta^* \times \sigma)r^{-1} = (\zeta^* \times \sigma)$ , and so a  $C$  satisfies  $C(S_0 \times \sigma_0) = (\eta^* \times \sigma)C$  if and only if  $rC$  satisfies the equation  $rC(S_0 \times \sigma_0) = r(\eta^* \times \sigma)r^{-1}(rC) = (\zeta^* \times \sigma)(rC)$ , i.e., if and only if  $rC$  lies in  $\mathcal{I}((S_0, \sigma_0), (\zeta^*, \sigma))$ . Thus if  $\eta$  and  $\zeta$  are similar, the spaces  $\bigcup\{Ran(C) \mid C \in \mathcal{I}((S_0, \sigma_0), (\eta^*, \sigma))\}$  and  $\bigcup\{Ran(C) \mid C \in \mathcal{I}((S_0, \sigma_0), (\zeta^*, \sigma))\}$  are identical. Similarly, if  $\eta$  and  $\zeta$  are similar, then the spaces  $\bigvee\{Ran(Q) \mid Q \in \sigma(M)', Q - \text{pure superharmonic for } \Phi_\eta\}$  and  $\bigvee\{Ran(Q) \mid Q \in \sigma(M)', Q - \text{pure superharmonic for } \Phi_\zeta\}$  are identical. Thus the theorem is an immediate consequence of Theorem 4.7.  $\square$

## 6 Induced Representations and their Ranges

In a sense, this section is an interlude that develops some ideas that will be used in the next section on the structure theorem. However, we believe the results in it are of sufficient interest in themselves that we want to develop them separately.

Throughout this section,  $\tau$  will be a normal representation of our  $W^*$ -algebra  $M$  on a Hilbert space  $H$  and  $\tau^{\mathcal{F}(E)}$  will be the induced representation of  $\mathcal{L}(\mathcal{F}(E))$  acting on the Hilbert space  $\mathcal{F}(E) \otimes_\tau H$ . The support projection of  $\tau$  will be denoted  $e$ . This is a central projection in  $M$  and  $e^\perp$  is the projection onto the kernel of  $\tau$ ,  $\ker(\tau)$ . The problem we want to address is this.

**Problem 6.1** *Determine when the image of  $H^\infty(E)$  under  $\tau^{\mathcal{F}(E)}$  is ultra-weakly closed.*

Of course,  $\tau^{\mathcal{F}(E)}$  is a normal representation of  $\mathcal{L}(\mathcal{F}(E))$  and so the image of  $\mathcal{L}(\mathcal{F}(E))$  in  $B(\mathcal{F}(E) \otimes_\tau H)$  is ultra-weakly closed, since  $\mathcal{L}(\mathcal{F}(E))$  is a  $W^*$ -algebra. Also, of course, if  $\tau$  is injective, then so is  $\tau^{\mathcal{F}(E)}$  and, consequently,  $\tau^{\mathcal{F}(E)}$  is isometric and an ultra-weak homeomorphism. In this event,  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  is an ultra-weakly closed subalgebra of  $B(\mathcal{F}(E) \otimes_\tau H)$ . In particular, if  $M$  is a factor, then  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  is ultra-weakly closed. The problem, then, is to determine what happens when the kernel of  $\tau$ ,  $e^\perp M$ , is non-trivial. In this case, the projection onto the kernel of  $\tau^{\mathcal{F}(E)}$  is  $I_{\mathcal{F}(E)} \otimes e^\perp$  and the problem is to see how it interacts with  $\tau^{\mathcal{F}(E)}(H^\infty(E))$ . We have no examples of representations  $\tau$  where the image  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  fails to be ultra-weakly closed, but we are able to provide useful, very general conditions on  $e$  that guarantee that  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  is ultra-weakly closed.

We adopt the following terminology, which is suggested by [7].

**Definition 6.2** A projection  $e$  in the center of  $M$ ,  $\mathfrak{Z}(M)$ , that satisfies  $\xi e = \varphi(e)\xi e$  for all  $\xi \in E$  will be called an  $E$ -saturated projection. If  $e$  also satisfies  $\xi e = \varphi(e)\xi$  for all  $\xi \in E$ ,  $e$  will be called an  $E$ -reducing projection.

**Example 6.3** If  $\alpha$  is an endomorphism of  $M$  and if  $E$  is the correspondence  ${}_{\alpha}M$ , then a central projection  $e \in M$  is  $E$ -saturated if and only if  $\alpha(e)ae = ae$  for all  $a \in M$ . That is,  $e$  is  $E$ -saturated if and only if  $e \leq \alpha(e)$ . Moreover,  $e$  will be  $E$ -reducing if and only if  $e$  is fixed by  $\alpha$ ,  $e = \alpha(e)$ .

The meaning the  $E$ -saturation condition for the present discussion may be further clarified by the following two lemmas.

**Lemma 6.4** A projection  $e$  in the center of  $M$ ,  $\mathfrak{Z}(M)$ , is an  $E$ -saturated projection in  $M$  if and only if  $\varphi_{\infty}(e)$  is an invariant projection for  $H^{\infty}(E)$  in the sense that

$$H^{\infty}(E)\varphi_{\infty}(e) = \varphi_{\infty}(e)H^{\infty}(E)\varphi_{\infty}(e). \quad (23)$$

Thus, if  $e$  is  $E$ -saturated, then in any completely contractive representation  $\rho$  of  $H^{\infty}(E)$ , the range of  $\rho(\varphi_{\infty}(e))$  is an invariant subspace  $\rho(H^{\infty}(E))$ .

**Proof.** First, recall that for  $a, b \in M$  and  $\xi \in E$ ,  $T_{\varphi(a)\xi b} = \varphi_{\infty}(a)T_{\xi}\varphi_{\infty}(b)$ . Consequently, if  $e$  is  $E$  saturated, so that by definition  $\xi e = \varphi(e)\xi e$  for all  $\xi \in E$ , it follows that for all  $\xi \in E$ ,  $T_{\xi}\varphi_{\infty}(e) = T_{\xi e}\varphi_{\infty}(e) = T_{\varphi(e)\xi e}\varphi_{\infty}(e) = \varphi_{\infty}(e)T_{\xi}\varphi_{\infty}(e)$ . Since  $\varphi_{\infty}(e)$  obviously commutes with  $\varphi_{\infty}(M)$ , equation 23 is verified. For the converse assertion, simply write out the matrices for  $T_{\xi}$ ,  $\xi \in E$ , and  $\varphi_{\infty}(e)$  with respect to the direct sum decomposition of  $\mathcal{F}(E) = \sum_{n \geq 0} E^{\otimes n}$  and compute what it means for the equation  $T_{\xi}\varphi_{\infty}(e) = \varphi_{\infty}(e)T_{\xi}\varphi_{\infty}(e)$  to hold.  $\square$

Note that the same argument shows  $e$  is  $E$ -reducing if and only if  $\varphi_{\infty}(e)$  commutes with  $H^{\infty}(E)$ . In fact, as we shall see in a moment, if  $e$  is  $E$ -reducing, then  $\varphi_{\infty}(e)$  lies in the center of  $\mathcal{L}(\mathcal{F}(E))$ .

**Lemma 6.5** If  $e$  is  $E$ -saturated, then the space  $Ee$  becomes a  $W^*$ -correspondence over  $Me$ . Moreover, the left action of  $M$  on  $E$  restricts to a unital left action of  $Me$  on  $Ee$ , and if we define  $\pi : Me \rightarrow \mathcal{L}(\mathcal{F}(E))$  and  $V : Ee \rightarrow \mathcal{L}(\mathcal{F}(E))$  by the formulae

$$\pi(me) := \varphi_{\infty}(me)$$



and

$$V(\xi e) := T_{\xi e},$$

then the pair  $(V, \pi)$  is an ultra-weakly continuous, isometric, covariant representation of  $(Ee, Me)$  in  $\mathcal{L}(\mathcal{F}(E))$ , whose image is contained in  $H^\infty(E)$ .

**Proof.** The calculation,

$$\begin{aligned} V(\xi e)^* V(\eta e) &= (T_\xi \varphi_\infty(e))^* T_\eta \varphi_\infty(e) = \varphi_\infty(e) T_\xi^* T_\eta \varphi_\infty(e) \\ &= \varphi_\infty(e) \varphi_\infty(\langle \xi, \eta \rangle) \varphi_\infty(e) = \pi(\langle \xi e, \eta e \rangle_{Me}), \end{aligned}$$

shows that  $V$  is isometric. The bimodule property is immediate. The ultra-weak continuity is an immediate consequence of [15, Lemma 2.5, Remark 2.6].  $\square$

**Remark 6.6** *Strictly speaking, of course,  $(V, \pi)$  is an isometric representation of  $(Ee, Me)$  into the abstract  $W^*$ -algebra,  $\mathcal{L}(\mathcal{F}(E))$ , so to apply the theory from [15] here and elsewhere, one should compose  $(V, \pi)$  with a faithful normal representation of  $\mathcal{L}(\mathcal{F}(E))$  on Hilbert space. The details are easy and may safely be omitted. Later, however, it will prove useful to use that device. Anticipating results to be proved shortly (Lemma 6.10), we call  $(V, \pi)$  or  $V \times \pi$  the canonical embedding of  $\mathcal{T}(Ee)$  in  $\mathcal{T}(E)$ . We will see that  $V \times \pi$  is faithful on the Toeplitz algebra,  $\mathcal{T}(Ee)$ , and extends to a completely isometric, ultra-weakly continuous representation of  $H^\infty(Ee)$ , mapping it into  $H^\infty(E)$ .*

The following lemma may be known, but we do not have a reference. It will be helpful to have the details in hand.

**Lemma 6.7** *Let  $F$  be a  $C^*$ -Hilbert module over a  $C^*$ -algebra  $N$ . For  $a \in N$ , define  $R_a : F \rightarrow F$  by the formula  $R_a \xi = \xi a$ ,  $\xi \in F$ . Then  $R_a$  is a bounded  $\mathbb{C}$ -linear operator on  $F$  with norm at most  $\|a\|$ . If  $a$  lies in the center of  $N$ ,  $\mathfrak{Z}(N)$ , then  $R_a$  is a bounded adjointable operator on  $F$  that lies in the center of  $\mathcal{L}(F)$ .*

**Proof.** For  $\xi \in F$  and  $a \in N$ , we have

$$\begin{aligned} \langle R_a \xi, R_a \xi \rangle &= \langle \xi a, \xi a \rangle \\ &= a^* \langle \xi, \xi \rangle a \leq a^* \|\xi\|^2 a \\ &\leq \|a\|^2 \|\xi\|^2, \end{aligned}$$

which shows that  $R_a$  is a continuous  $\mathbb{C}$ -linear operator with norm bounded by  $\|a\|$ . To see that  $R_a \in \mathcal{L}(F)$  when  $a \in \mathfrak{Z}(N)$ , simply observe that for  $\xi$  and  $\eta$  in  $F$ ,

$$\langle R_a \xi, \eta \rangle = \langle \xi a, \eta \rangle = a^* \langle \xi, \eta \rangle = \langle \xi, \eta \rangle a^* = \langle \xi, \eta a^* \rangle = \langle \xi, R_{a^*} \eta \rangle.$$

This shows that  $R_a$  is adjointable, with adjoint  $R_{a^*}$  and this, in turn, shows that  $R_a$  is  $N$ -linear. Thus,  $R_a \in \mathcal{L}(F)$ . (Of course, the fact that  $a$  lies in  $\mathfrak{Z}(N)$  also implies directly that  $R_a$  is  $N$ -linear.) However, since elements of  $\mathcal{L}(F)$  are  $N$ -module maps, i.e.,  $T(\xi b) = (T\xi)b$ , it is immediate that  $R_a$  lies in the center of  $\mathcal{L}(F)$ .  $\square$

Among other things, the following lemma solves Problem 6.1 under the hypothesis that the support projection of the representation  $\tau$  is  $E$ -reducing.

**Lemma 6.8** *Let  $\tau$  be a normal representation of the  $W^*$ -algebra  $M$  on a Hilbert space  $K$  and let  $e$  be its support projection. Let  $q$  be the smallest projection in  $M$  such that  $\varphi_\infty(q)R_e = R_e$ . Then the following assertions hold:*

$$(1) \ker(\tau^{\mathcal{F}(E)}) = \{R \in \mathcal{L}(\mathcal{F}(E)) \mid RR_e = 0\}.$$

(2) *The ultra-weakly closed ideal*

$$\begin{aligned} H^\infty(E) \cap \ker(\tau^{\mathcal{F}(E)}) &= \{R \in H^\infty(E) : R\varphi_\infty(q) = 0\} \\ &= \overline{H^\infty(E)\varphi_\infty(q^\perp)H^\infty(E)}^{u-w}, \end{aligned}$$

*in  $H^\infty(E)$  is generated by  $\varphi_\infty(q^\perp)$ .*

(3) *The subspace  $\varphi_\infty(q)\mathcal{F}(E)$  of  $\mathcal{F}(E)$  is invariant for  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  and the map  $X \rightarrow \tau^{\mathcal{F}(E)}(X)|_{\varphi_\infty(q)\mathcal{F}(E)}$  is an injective completely contractive representation of  $H^\infty(E)$ .*

(4) *If  $e$  is  $E$ -saturated then  $e = q$ , so that  $e$  is  $\mathcal{F}(E)$ -saturated, i.e.,  $\varphi_\infty(e)\mathcal{F}(E)e = \mathcal{F}(E)e$ .*

(5) *If  $e$  is  $E$ -reducing, the three projections  $R_e$ ,  $\varphi_\infty(e)$ , and  $\varphi_\infty(q)$  coincide and lie in the center of  $\mathcal{L}(\mathcal{F}(E))$ . Consequently,  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  is ultra-weakly closed and the restriction of  $\tau^{\mathcal{F}(E)}$  to  $H^\infty(E)\varphi_\infty(e)$  is completely isometric.*

**Proof.** For  $R \in \mathcal{L}(\mathcal{F}(E))$ ,  $R \otimes_\tau I_K = 0$  if and only if for every  $k \in K$  and  $\eta \in \mathcal{F}(E)$ ,  $0 = \langle R\eta \otimes k, R\eta \otimes k \rangle = \langle k, \tau(\langle R\eta, R\eta \rangle)k \rangle$ , that is, if and only if  $\langle R\eta, R\eta \rangle \in Me^\perp$ . This happens if and only if  $R = RR_e^\perp$ . It follows that

$$\ker(\tau^{\mathcal{F}(E)}) = \{R \in \mathcal{L}(\mathcal{F}(E)) \mid RR_e = 0\}$$

and

$$H^\infty(E) \cap \ker(\tau^{\mathcal{F}(E)}) = \{R \in H^\infty(E) \mid RR_e = 0\}. \quad (24)$$

Now choose a faithful normal representation,  $\sigma$ , of  $M$  on a Hilbert space  $H$ . Then  $\sigma^{\mathcal{F}(E)}$  is a \*-isomorphism of  $\mathcal{L}(\mathcal{F}(E))$  onto  $\mathcal{L}(\mathcal{F}(E)) \otimes I_H$  (and is therefore also a homeomorphism with respect to the ultra-weak topologies.) Also note that  $I_{\mathcal{F}(E)} \otimes \sigma(e) = R_e \otimes I_H = \sigma^{\mathcal{F}(E)}(R_e)$  by Lemma 6.7. Set

$$g = \bigvee \{u(\sigma^{\mathcal{F}(E)}(R_e))u^* \mid u \in \sigma^{\mathcal{F}(E)}(\varphi_\infty(M))', u \text{ is unitary}\}.$$

The range of the projection  $u(\sigma^{\mathcal{F}(E)}(R_e))u^*$  is  $u(R_e\mathcal{F}(E) \otimes_\sigma H)$  and, thus,  $\varphi_\infty(q^\perp) \otimes I_H = \sigma^{\mathcal{F}(E)}(\varphi_\infty(q^\perp))$  vanishes on it. It follows that  $g \leq \sigma^{\mathcal{F}(E)}(\varphi_\infty(q))$ . By construction  $g$  is a projection in the center of  $\sigma^{\mathcal{F}(E)}(\varphi_\infty(M))$ , so we can write  $g = \sigma^{\mathcal{F}(E)}(\varphi_\infty(z))$  for some projection  $z \in \mathfrak{Z}(M)$ . But then  $\varphi_\infty(q - z)$  vanishes on  $R_e\mathcal{F}(E)$ , which implies that  $q = z$ , since  $q$  is the smallest projection in  $M$  with this property. Consequently,

$$\sigma^{\mathcal{F}(E)}(\varphi_\infty(q)) = \bigvee \{u(\sigma^{\mathcal{F}(E)}(R_e))u^* \mid u \in \sigma^{\mathcal{F}(E)}(\varphi_\infty(M))', u \text{ is unitary}\}, \quad (25)$$

and  $q \in \mathfrak{Z}(M)$ .

Next we want to show that if  $R \in H^\infty(E)$  and if  $RR_e = 0$ , then  $R\varphi_\infty(q) = 0$ . To this end, we use the gauge automorphism group and the notation developed in paragraph 2.9. Observe that  $W_t$  commutes with  $R_e$  for all  $t \in \mathbb{T}$ , since  $W_t \in \mathcal{L}(\mathcal{F}(E))$  and  $R_e \in \mathfrak{Z}(\mathcal{L}(\mathcal{F}(E)))$ . Consequently,  $\gamma_t(R)R_e = W_t R R_e W_t^* = 0$ , and so  $\Phi_k(R)R_e = 0$  for all  $k$ , by equation (8). Since  $\Phi_k(R)^* \Phi_k(R) \in \varphi_\infty(M)$ , equation (25) implies that  $\Phi_k(R)\varphi_\infty(q) = 0$  for every  $k$ . Consequently each of the Cesaro sums  $\Sigma_n(R) := \sum_{0 \leq j < n} (1 - \frac{j}{n}) \Phi_j(R)$  satisfies the equation  $\Sigma_n(R)\varphi_\infty(q) = 0$ ,  $n \geq 0$ . Since  $R$  is the ultra-weak limit of the Cesaro sums,  $\sum_n(R)$ , we conclude that  $R\varphi_\infty(q) = 0$ , as we wanted to show.

This argument shows, too, that the map sending  $R\varphi_\infty(q)$  to  $RR_e$ , for  $R \in H^\infty(E)$ , is injective. It is also completely contractive and ultra-weakly continuous.

Further, it is clear from equation (24) that

$$H^\infty(E) \cap \ker(\tau^{\mathcal{F}(E)}) = \{R \in H^\infty(E) : R\varphi_\infty(q) = 0\}.$$

Thus  $\varphi_\infty(q^\perp)$  lies in the kernel of  $\tau^{\mathcal{F}(E)}$ , and the ultra-weakly closed ideal in  $H^\infty(E)$  generated by  $\varphi_\infty(q^\perp)$  lies in the kernel as well. On the other hand, every  $R$  in the kernel satisfies  $R = R\varphi_\infty(q^\perp)$  and, thus, is contained in the ideal generated by  $\varphi_\infty(q^\perp)$ . It follows that  $\varphi_\infty(q^\perp)H^\infty(E) \subseteq \ker(\tau^{\mathcal{F}(E)}) \cap H^\infty(E) = \{R \in H^\infty(E) : R\varphi_\infty(q) = 0\}$  and, consequently,

$$\varphi_\infty(q^\perp)H^\infty(E)\varphi_\infty(q) = \{0\},$$

that  $e$  is  $E$ -saturated and is proved in i.e.,  $\varphi_\infty(q)$  is an invariant projection for  $H^\infty(E)$ .

The restriction of  $\tau^{\mathcal{F}(E)}$  to  $R_e\mathcal{L}(\mathcal{F}(E))R_e = \mathcal{L}(\mathcal{F}(E))R_e$  is an isomorphism of this von Neumann algebra onto the image of the induced representation. It is, therefore, completely isometric. The restriction of the induced representation to  $\varphi_\infty(q)H^\infty(E)\varphi_\infty(q)$  is a composition of two completely contractive injective maps and is, therefore, completely contractive and injective.

For assertion (5), note that, if  $e$  is  $E$ -saturated then, for every  $n \leq 1$  and every  $\xi \in E^{\otimes n}$ ,  $\xi e = \varphi_n(e)\xi e$ . Thus  $\mathcal{F}(E)e = \varphi_\infty(e)\mathcal{F}(E)e$  and so  $q = e$ .

For assertion (6), assume that  $e$  is  $E$ -reducing. Then it is easy to verify that  $R_e = \varphi_\infty(e)$  and, in particular,  $R_e$  lies in the center of  $H^\infty(E)$ . The map  $\tau^{\mathcal{F}(E)}$ , restricted to the von Neumann algebra  $\mathcal{L}(\mathcal{F}(E))R_e$ , is an injective ultra-weakly continuous representation and, thus, is a homeomorphism with respect to the ultra-weak topologies onto its image. Consequently, its restriction to  $H^\infty(E)\varphi_\infty(e)$  is completely isometric and has a ultra-weakly closed image which is  $\tau^{\mathcal{F}(E)}(H^\infty(E))$ . This proves (6).  $\square$

The following Theorem solves Problem 6.1 under the hypothesis that the support projection is saturated.

**Theorem 6.9** *If the support projection  $e$  of a normal representation  $\tau$  of  $M$  is  $E$ -saturated, then  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  is ultra-weakly closed and the restriction of  $\tau^{\mathcal{F}(E)}$  to  $H^\infty(E)\varphi_\infty(e)$  is completely isometric.*

We require two lemmas.

**Lemma 6.10** *If  $e$  is an  $E$ -saturated central projection in  $M$ , then the canonical embedding  $V \times \pi$  of  $\mathcal{T}(Ee)$  into  $\mathcal{T}(E)$ , defined in Lemma 6.5, is faithful.*

**Proof.** To show  $V \times \pi$  is faithful, it suffices to prove that if  $\sigma_0$  is a faithful normal representation of  $M$  on the Hilbert space  $H$  and if  $\pi_0 = \sigma_0^{\mathcal{F}(E)} \circ \pi$  and if  $V_0 = \sigma_0^{\mathcal{F}(E)} \circ V$ , then  $V_0 \times \pi_0$  is a faithful representation of  $\mathcal{T}(Ee)$  on  $\mathcal{F}(E) \otimes_{\sigma_0} H$ . This is because  $\sigma_0^{\mathcal{F}(E)}$  is a faithful normal representation of  $\mathcal{L}(\mathcal{F}(E))$ . We use [8, Theorem 2.1], which asserts in our setting, that  $V_0 \times \pi_0$  will be faithful if the representation of  $Me$  obtained by restricting  $\pi_0(Me)$  to  $(\mathcal{F}(E) \otimes_{\sigma_0} H) \ominus (V_0(Ee)(\mathcal{F}(E) \otimes_{\sigma_0} H))$  is faithful. For this, observe that because  $e$  is  $E$ -saturated

$$\begin{aligned} V_0(Ee)(\mathcal{F}(E) \otimes_{\sigma_0} H) &= \sigma_0^{\mathcal{F}(E)}(V(Ee))(\mathcal{F}(E) \otimes_{\sigma_0} H) \\ &= Ee \otimes \mathcal{F}(E) \otimes_{\sigma_0} H = E \otimes \varphi_{\infty}(e)\mathcal{F}(E) \otimes_{\sigma_0} H \\ &= E \otimes \mathcal{F}(E)e \otimes_{\sigma_0} H = E \otimes \mathcal{F}(E) \otimes \sigma_0(e)H, \end{aligned}$$

by point (4) of Lemma 6.8. Since  $E \otimes \mathcal{F}(E) = \sum_{n \geq 1} E^{\otimes n}$ , we see that  $(\mathcal{F}(E) \otimes_{\sigma_0} H) \ominus (V_0(Ee)(\mathcal{F}(E) \otimes_{\sigma_0} H))$  contains  $M \otimes_{\sigma_0} \sigma_0(e)H$  as a summand, to which  $\pi_0(Me)$  restricts. But that restriction is obviously faithful on  $Me$  since it is just left multiplication on  $M \otimes_{\sigma_0} \sigma_0(e)H = Me \otimes_{\sigma_0} H$  by elements of the form  $me \otimes I_H$ ,  $m \in M$ , and  $\sigma_0$  is faithful.  $\square$

**Lemma 6.11** *Suppose  $\tau$  is a normal representation of  $M$ , suppose its support projection  $e$  is  $E$ -saturated, and let  $\tau_e$  be the restriction of  $\tau$  to  $Me$ . Then the map that sends  $\xi e \otimes_{\tau_e} k$  in  $\mathcal{F}(Ee) \otimes_{\tau_e} K$  to  $\xi \otimes_{\tau} k$  in  $\mathcal{F}(E) \otimes_{\tau} K$  extends to a well-defined Hilbert space isomorphism  $U$  mapping  $\mathcal{F}(Ee) \otimes_{\tau_e} K$  onto  $\mathcal{F}(E) \otimes_{\tau} K$  such that, for every  $X \in \mathcal{T}_+(Ee)$ ,*

$$\tau_e^{\mathcal{F}(Ee)}(X) = U^* \tau^{\mathcal{F}(E)}((V \times \pi)(X))U,$$

where  $(V, \pi)$  is the canonical embedding of  $(Ee, Me)$  in  $\mathcal{T}(E)$ . The image of  $\mathcal{T}_+(Ee)$  under  $V \times \pi$  is  $\mathcal{T}_+(E)\varphi_{\infty}(e) = \varphi_{\infty}(e)\mathcal{T}_+(E)\varphi_{\infty}(e)$ .

**Proof.** The fact that  $U$  is well-defined and isometric is an easy calculation. The fact that  $U$  is surjective is immediate. Observe that  $\tau_e^{\mathcal{F}(Ee)}(T_{\theta e})(\xi e \otimes_{\tau_e} k) = \theta e \otimes \xi e \otimes_{\tau_e} k$ . By definition of  $U$ , this last expression is

$$\begin{aligned} U^*(\theta e \otimes (\xi \otimes_{\tau} k)) &= U^*(\theta \otimes \varphi_{\infty}(e)\xi \otimes_{\tau} k) = U^* \tau^{\mathcal{F}(E)}(T_{\theta})(\varphi_{\infty}(e)\xi \otimes_{\tau} k) \\ &= U^* \tau^{\mathcal{F}(E)}(T_{\theta} \varphi_{\infty}(e))U(\xi e \otimes_{\tau_e} k) \\ &= U^* \tau^{\mathcal{F}(E)}((V \times \pi)(T_{\theta e}))U(\xi e \otimes_{\tau_e} k). \end{aligned}$$

Similarly,  $\tau_e^{\mathcal{F}(Ee)}(\varphi_\infty^e(me)) = U^* \tau^{\mathcal{F}(E)}((V \times \pi)(\varphi_\infty(me))U$  for all  $me \in Me$ , where  $\varphi_\infty^e$  denotes the action of  $Me$  on  $\mathcal{F}(Ee)$ .  $\square$

**Proof of Theorem 6.9.** Since  $\tau_e$  is a faithful representation of  $Me$ ,  $\tau_e^{\mathcal{F}(E)}$  is a completely isometric map of  $H^\infty(Ee)$  and its image,  $\tau_e^{\mathcal{F}(E)}(H^\infty(Ee))$ , is ultra-weakly closed. Since both  $\tau^{\mathcal{F}(E)}$  and  $\tau_e^{\mathcal{F}(E)}$  are ultra-weakly continuous, we have

$$\overline{\tau^{\mathcal{F}(E)}(\mathcal{T}_+(E))}^{u-w} = \overline{\tau^{\mathcal{F}(E)}(H^\infty(E))}^{u-w}$$

and

$$\overline{\tau_e^{\mathcal{F}(Ee)}(\mathcal{T}_+(Ee))}^{u-w} = \overline{\tau_e^{\mathcal{F}(Ee)}(H^\infty(Ee))}^{u-w} = \tau_e^{\mathcal{F}(Ee)}(H^\infty(Ee)).$$

Now  $\overline{\tau^{\mathcal{F}(E)}(H^\infty(E))}^{u-w} = \overline{\tau^{\mathcal{F}(E)}(H^\infty(E)\varphi_\infty(e))}^{u-w}$  since  $\varphi_\infty(e^\perp)$  is the projection onto the kernel of  $\tau^{\mathcal{F}(E)}$ . Also, Lemma 6.11 implies that  $(V \times \pi)(\mathcal{T}_+(Ee)) = \mathcal{T}_+(E)\varphi_\infty(e)$ . So  $\overline{(V \times \pi)(\mathcal{T}_+(Ee))}^{u-w} = H^\infty(E)\varphi_\infty(e)$ . Consequently, from Lemma 6.11, we conclude that  $\overline{\tau^{\mathcal{F}(E)}(H^\infty(E)\varphi_\infty(e))}^{u-w} = U \tau_e^{\mathcal{F}(Ee)}(H^\infty(Ee)) U^*$ .

Thus, given  $Z \in \overline{\tau^{\mathcal{F}(E)}(H^\infty(E))}^{u-w}$ , there is a net  $\{X_n\}$  of elements of  $\mathcal{T}_+(Ee)$  that converges ultra-weakly to  $X \in H^\infty(Ee)$  such that  $\|X_n\| \leq \|X\|$  for all  $n$  and  $Z = U \tau_e^{\mathcal{F}(Ee)}(X) U^*$ . The net  $\{(V \times \pi)(X_n)\}$  is bounded in  $\mathcal{T}_+(E)\varphi_\infty(e)$  and so has a subnet  $\{(V \times \pi)(X_{n_\alpha})\}$  that converges ultra-weakly to some  $Y \in H^\infty(E)\varphi_\infty(e)$ , with  $\|Y\| \leq \|X\|$ . However, from the ultra-weak continuity of the maps  $\tau^{\mathcal{F}(E)}$  and  $\tau_e^{\mathcal{F}(Ee)}$ , we conclude that

$$\begin{aligned} Z &= U \tau_e^{\mathcal{F}(Ee)}(X) U^* = \lim U \tau_e^{\mathcal{F}(Ee)}(X_{n_\alpha}) U^* = \lim \tau^{\mathcal{F}(E)}(X_{n_\alpha}) \\ &= \tau^{\mathcal{F}(E)}(Y) \end{aligned}$$

belongs to  $\tau^{\mathcal{F}(E)}(H^\infty(E)\varphi_\infty(e))$ . This shows that  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  is ultra-weakly closed. We also have  $\|Y\| \leq \|X\| = \|Z\| \leq \|\tau^{\mathcal{F}(E)}(Y)\| \leq \|Y\|$ . Thus  $\|Y\| = \|\tau^{\mathcal{F}(E)}(Y)\|$ . If we started with a given  $Y \in H^\infty(E)\varphi_\infty(e)$  and  $Z = \tau^{\mathcal{F}(E)}(Y)$ , we would be able to conclude that  $\|Y\| = \|Z\|$  so that the map  $\tau^{\mathcal{F}(E)}$ , restricted to  $H^\infty(E)\varphi_\infty(e)$ , is isometric. One can argue similarly to show that it is completely isometric.  $\square$

## 7 The Structure Theorem

We have two goals in this section. The first is Theorem 7.8, which is a generalization of [2, Theorem 2.6], that Davidson, Katsoulis and Pitts call the *Structure Theorem*.

Suppose, then, that  $(S, \sigma)$  is an isometric representation of  $(E, M)$  acting on a Hilbert space  $H$ . We shall write  $\mathcal{S}$  for the ultra-weak closure of  $(S \times \sigma)(\mathcal{T}_+(E))$ . It is thus the ultra-weakly closed algebra generated by the operators  $\{\sigma(a), S(\xi) \mid \xi \in E, a \in M\}$ . Also, we shall write  $\mathcal{S}_0$  for the ultra-weakly closed algebra generated by  $\{S(\xi) \mid \xi \in E\}$ . Evidently,  $\mathcal{S}_0$  is an ultra-weakly closed, two-sided ideal in  $\mathcal{S}$ . The decomposition that Davidson, Katsoulis and Pitts advanced centers on understanding the position of  $\mathcal{S}_0$  in  $\mathcal{S}$ . In particular, it is important to know when  $\mathcal{S}_0$  is a proper ideal of  $\mathcal{S}$ . Our analysis follows a similar route, but it is made more complicated by the presence of  $M$ . Observe that since  $\mathcal{S}_0$  is an ultra-weakly closed 2-sided ideal in  $\mathcal{S}$ ,  $\sigma^{-1}(\mathcal{S}_0)$  is an ultra-weakly closed two sided ideal in  $M$  and hence is of the form  $pM$  for a suitable projection in the center of  $M$ .

**Definition 7.1** *The projection  $e$  in  $M$  with the property that  $\sigma^{-1}(\mathcal{S}_0) = e^\perp M$  is called the model projection for  $\mathcal{S}$  (or for  $(S, \sigma)$ ).*

The reason for terminology will become clear through a brief outline for our analysis that we hope will be helpful when following our arguments. As in [2], we let  $N$  be the von Neumann algebra generated by  $\mathcal{S}$ . We will see in Proposition 7.5 that  $\bigcap_{k=1}^\infty \mathcal{S}_0^k$  is a left ideal in  $N$ . Consequently, there is a projection  $P \in \mathcal{S} \cap \sigma(M)'$  such that  $\bigcap_{k=1}^\infty \mathcal{S}_0^k = NP = SP$ . Following [2] we call this projection *the structure projection* for  $\mathcal{S}$ . We will show in Theorem 7.8 that  $(I - P)H = P^\perp H$  is invariant under  $\mathcal{S}$  and that  $\mathcal{S} = NP + P^\perp \mathcal{S} P^\perp$ . That is, in matrix form,

$$\mathcal{S} = \begin{bmatrix} PNP & 0 \\ P^\perp NP & P^\perp \mathcal{S} P^\perp \end{bmatrix}.$$

We will see in Lemma 7.2 that  $e$  is  $E$ -saturated and that it is the support projection for an induced representation  $\tau^{\mathcal{F}(E)}$  that we will soon describe. We will see in Lemma 7.6 that  $\sigma(e)$  is the central support of  $P^\perp$  in  $\sigma(M)$ . In part (3) of Theorem 7.8 we will see that  $P^\perp \mathcal{S} P^\perp$  is completely isometrically isomorphic and ultra-weakly homeomorphic to  $\tau^{\mathcal{F}(E)}(H^\infty(E))$ . Thus  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  serves as a model for  $P^\perp \mathcal{S} P^\perp$ . Further, it will follow from

Theorem 7.9 that if  $\sigma(e^\perp) = P = 0$ , then the representation  $(S, \sigma)$  is absolutely continuous.

Concerning this last statement, one might be inclined at first to believe that once one knows that  $\mathcal{S}$  is completely isometrically isomorphic and ultra-weakly homeomorphic to the ultra-weak closure of the range of an induced representation, then  $(S, \sigma)$ , itself, must be an induced representation, but examples constructed by Davidson, Katsoulis and Pitts in [2, Section 3] show this is not the case. Thus our Theorem 7.9 leads to a much more inclusive result and one may wonder about a converse: Does every absolutely continuous, isometric representation  $(S, \sigma)$  have a vanishing structure projection? There answer, in general, is “no”. Indeed, even in the case when  $A = \mathbb{C} = E$ , the answer is “no” for classical reasons. In this situation, it follows from Szegő’s theorem that if  $S(1)$  is an absolutely continuous unitary operator whose spectrum does not cover the circle, then  $P = I$ . It is of interest to determine more precisely when this may happen in the general setting.

To proceed with the details, we fix the isometric representation  $(S, \sigma)$  and we begin by analyzing the model projection  $e$ . For this purpose we observe that since we are working with ultra-weakly closed algebras, ultra-weakly closed ideals and related constructs involving the ultra-weak topology, it will be convenient to employ an infinite multiple  $(S^{(\infty)}, \sigma^{(\infty)})$  of  $(S, \sigma)$ , which acts on the countably infinite direct sum of copies of  $H$ ,  $H^{(\infty)}$ . In general, for  $x \in B(H)$ , we write  $x^{(\infty)}$  for its infinite ampliation in  $B(H^{(\infty)})$ . Likewise for a space of operators  $X \subseteq B(H)$ , we write  $X^{(\infty)}$  for  $\{x^{(\infty)} \mid x \in X\}$ . Note that for  $x \in H^\infty(E)$ , we have  $S^{(\infty)} \times \sigma^{(\infty)}(x) = (S \times \sigma(x))^\infty$ . In particular,  $S^{(\infty)}(\xi) = S(\xi)^{(\infty)}$ , for all  $\xi \in E$ .

We are especially interested in the set  $\mathfrak{M}$  that consists of all subspaces  $\mathcal{M}$  of  $H^{(\infty)}$  that reduce  $\sigma^{(\infty)}(M)$  and are *wandering subspaces* for  $S^{(\infty)}$  in the sense that  $\mathcal{M}$ ,  $[S^{(\infty)}(E)\mathcal{M}]$ ,  $[S^{(\infty)}(E)S^{(\infty)}(E)\mathcal{M}]$ , etc. are all orthogonal, and their direct sum is a subspace of  $H^{(\infty)}$  that is invariant for the algebra  $S^{(\infty)} \times \sigma^{(\infty)}(H^\infty(E))$ . This subspace must be invariant for the ultra-weak closure of  $S^{(\infty)} \times \sigma^{(\infty)}(H^\infty(E))$ , which is the same as  $\mathcal{S}^{(\infty)}$ . It is easily seen, then, that  $\mathcal{M} \oplus [S^{(\infty)}(E)\mathcal{M}] \oplus [S^{(\infty)}(E)S^{(\infty)}(E)\mathcal{M}] \oplus \dots = [\mathcal{S}^{(\infty)}(\mathcal{M})]$ . For  $\mathcal{M}$  in  $\mathfrak{M}$ , we denote the restriction of  $(S^{(\infty)}, \sigma^{(\infty)})$  to  $[\mathcal{S}^{(\infty)}(\mathcal{M})]$  by  $(S_{\mathcal{M}}^{(\infty)}, \sigma_{\mathcal{M}}^{(\infty)})$ . Evidently,  $(S_{\mathcal{M}}^{(\infty)}, \sigma_{\mathcal{M}}^{(\infty)})$  is unitarily equivalent to the induced representation  $\tau_{\mathcal{M}}^{\mathcal{F}(E)}$  where  $\tau_{\mathcal{M}}$  is the restriction of  $\sigma^{(\infty)}$  to  $\mathcal{M}$ . In more detail, since  $\mathcal{M} \oplus [S^{(\infty)}(E)\mathcal{M}] \oplus [S^{(\infty)}(E)S^{(\infty)}(E)\mathcal{M}] \oplus \dots = [\mathcal{S}^{(\infty)}(\mathcal{M})]$ , the map which takes  $h_0 \oplus S^{(\infty)}(\xi_1)h_1 \oplus S^{(\infty)}(\xi_{21})S^{(\infty)}(\xi_{22})h_2 \oplus \dots \oplus S^{(\infty)}(\xi_{n1})S^{(\infty)}(\xi_{n2}) \dots S^{(\infty)}(\xi_{nn})h_n$



in  $[\mathcal{S}^{(\infty)}(\mathcal{M})]$  to  $h_0 \oplus \xi_1 \otimes h_1 \oplus \xi_{21} \otimes \xi_{22} \otimes h_2 \oplus \cdots \oplus \xi_{n1} \otimes \xi_{n2} \otimes \cdots \otimes \xi_{nn} \otimes h_n$  in  $\mathcal{F}(E) \otimes_{\tau_{\mathcal{M}}} \mathcal{M}$  is well defined and extends to a Hilbert space isomorphism  $U_{\mathcal{M}} : [\mathcal{S}^{(\infty)}(\mathcal{M})] \rightarrow \mathcal{F}(E) \otimes_{\tau_{\mathcal{M}}} \mathcal{M}$  such that

$$U_{\mathcal{M}}^*(\varphi_{\infty}(a) \otimes I_{\mathcal{M}})U_{\mathcal{M}} = \sigma^{(\infty)}(a)|[\mathcal{S}^{(\infty)}\mathcal{M}] , \quad a \in M,$$

and

$$U_{\mathcal{M}}^*(T_{\xi} \otimes I_{\mathcal{M}})U_{\mathcal{M}} = S^{(\infty)}(\xi)|[\mathcal{S}^{(\infty)}\mathcal{M}] , \quad \xi \in E.$$

(See [15, Remark 7.7]).

Although it leads to non-separable spaces, in general, and introduces a lot of redundancy into our analysis, it is convenient to let  $\mathcal{E}$  be the external direct sum of all the spaces  $\mathcal{M}$  in  $\mathfrak{M}$ , writing

$$\mathcal{E} := \sum_{\mathcal{M} \in \mathfrak{M}}^{\oplus} \mathcal{M},$$

and setting

$$\tau := \sum_{\mathcal{M} \in \mathfrak{M}}^{\oplus} \tau_{\mathcal{M}}. \tag{26}$$

Then  $\tau^{\mathcal{F}(E)}$  gives representations of  $\mathcal{T}_+(E)$  and  $H^{\infty}(E)$  on  $\mathcal{F}(E) \otimes_{\tau} \mathcal{E}$ . Further, if we write  $U := \sum \oplus U_{\mathcal{M}}$ , then  $U$  implements a unitary equivalence between  $\tau^{\mathcal{F}(E)}$  and  $\sum_{\mathcal{M} \in \mathfrak{M}} \oplus (S_{\mathcal{M}}^{(\infty)} \times \sigma_{\mathcal{M}}^{(\infty)}) = \sum_{\mathcal{M} \in \mathfrak{M}} \oplus (S^{(\infty)} \times \sigma^{(\infty)})|[\mathcal{S}^{(\infty)}\mathcal{M}]$ . The map

$$\Phi : B(H) \rightarrow B(\mathcal{F}(E) \otimes_{\tau} \mathcal{E})$$

defined by the formula

$$\Phi(x) = U(\sum \oplus P_{[\mathcal{S}^{(\infty)}\mathcal{M}]}x^{(\infty)}|[\mathcal{S}^{(\infty)}\mathcal{M}])U^*, \tag{27}$$

where  $P_{[\mathcal{S}^{(\infty)}\mathcal{M}]}$  is the projection of  $H^{(\infty)}$  onto  $[\mathcal{S}^{(\infty)}\mathcal{M}]$ , is a completely positive map that is continuous with respect to the ultra-weak topologies. When restricted to  $\mathcal{S}$ ,

$$\Phi(x) = U(\sum \oplus x^{(\infty)}|[\mathcal{S}^{(\infty)}\mathcal{M}])U^* , \quad x \in \mathcal{S}, \tag{28}$$

so this restriction, also denoted  $\Phi$ , is a homomorphism that satisfies  $\Phi(\sigma(a)) = \varphi_{\infty}(a)$  for  $a \in M$  and  $\Phi(S(\xi)) = T_{\xi} \otimes I_{\mathcal{E}}$  for  $\xi \in E$ . Thus  $\Phi(\mathcal{S})$  is ultra-weakly

dense in the ultra-weak closure of  $\tau^{\mathcal{F}(E)}(\mathcal{T}_+(E))$ . (Note that each element in this ultra-weak closure can be approximated by Cesaro means calculated with respect to the gauge automorphism group, and each such mean lies in  $\Phi(\mathcal{S})$ ).

**Lemma 7.2** *The model projection  $e$  for  $\mathcal{S}$  is the support projection of  $\tau$  and is  $E$ -saturated.*

**Proof.** Recall that by definition,  $e^\perp$  is the central cover of  $\{a \in M \mid \sigma(a) \in \mathcal{S}_0\}$ , which is an ultra-weakly closed ideal in  $M$ . Since  $\ker(\tau)$  is also an ultra-weakly closed ideal in  $M$ , it suffices to show that each contains the same projections  $p$ . If  $\sigma(p) \in \mathcal{S}_0$ , then for  $\mathcal{M} \in \mathfrak{M}$ ,  $\tau_{\mathcal{M}}(p)\mathcal{M} = \sigma^{(\infty)}(p)\mathcal{M} \subseteq \mathcal{S}_0(\mathcal{M}) \subseteq \mathcal{M}^\perp$ . But  $\tau_{\mathcal{M}}(p)\mathcal{M} \subseteq \mathcal{M}$ . Thus  $\tau_{\mathcal{M}}(p) = 0$ . Since  $\tau$  is defined to be  $\sum_{\mathcal{M} \in \mathfrak{M}} \tau_{\mathcal{M}}$ ,  $p \in \ker(\tau)$ . On the other hand, if  $\sigma(p)$  is not in  $\mathcal{S}_0$ , then there is an ultra-weakly continuous linear functional  $f$  such that  $f(\sigma(p)) \neq 0$  and  $f(a) = 0$  for  $a \in \mathcal{S}_0$ . But then we can write  $f(x) = \langle x^{(\infty)}\xi, \eta \rangle$  for suitable vectors  $\xi, \eta \in H^{(\infty)}$ , and find that  $\eta$  is orthogonal to the  $\sigma^{(\infty)}(M)$ -invariant subspace  $[(\mathcal{S}_0)^{(\infty)}\xi]$ , on the one hand, but is not orthogonal to  $\sigma^{(\infty)}(p)\xi$ , on the other. Consequently, the space  $\mathcal{M} := [\sigma^{(\infty)}(p)\mathcal{S}^{(\infty)}\xi] \ominus [\sigma^{(\infty)}(p)\mathcal{S}_0^{(\infty)}\xi]$  is non zero, and by construction,  $\mathcal{M}$  is a  $\sigma^{(\infty)}(M)$ -invariant wandering subspace. That is,  $\mathcal{M}$  lies in  $\mathfrak{M}$ , and evidently,  $\tau_{\mathcal{M}}(p) = I_{\mathcal{M}} \neq 0$ . Thus  $\ker(\tau) = \{a \in M \mid \sigma(a) \in \mathcal{S}_0\}$ .

To see that  $e$  is  $E$ -saturated, i.e., to see that  $\varphi(e^\perp)\xi e = 0$  for all  $\xi \in E$ , note that  $\sigma(e^\perp)$  lies in  $\mathcal{S}_0$ . Thus, for every  $\xi, \eta \in E$ ,  $S(\eta)^*\sigma(e^\perp)S(\xi) \in \mathcal{S}_0$ . But  $S(\eta)^*\sigma(e^\perp)S(\xi) = \sigma(\langle \eta, \varphi(e^\perp)\xi \rangle)$  and so it follows from the definition of  $e$  that  $\langle \eta, \varphi(e^\perp)\xi e \rangle = \langle \eta, \varphi(e^\perp)\xi \rangle e = 0$  for all  $\xi, \eta \in E$ . Thus  $\varphi(e^\perp)\xi e = 0$  for all  $\xi \in E$ .  $\square$

The representation  $\tau$  acts on a non-separable space, so it may be comforting to know that for all intents and purposes we have in mind, it is possible to replace  $\tau$  with a representation on a separable space. This an immediate consequence of the following proposition.

**Proposition 7.3** *Suppose  $\tau_1$  and  $\tau_2$  are two normal representations of  $M$  on Hilbert spaces  $H_1$  and  $H_2$ , respectively, and suppose that  $\ker(\tau_1) = \ker(\tau_2)$ . Then the map  $\Psi$  that sends  $\tau_1^{\mathcal{F}(E)}(R)$  to  $\tau_2^{\mathcal{F}(E)}(R)$ ,  $R \in \mathcal{L}(\mathcal{F}(E))$ , is a normal  $*$ -isomorphism from the von Neumann algebra  $\mathcal{L}(\mathcal{F}(E)) \otimes_{\tau_1} I_{H_1}$  onto the von Neumann algebra  $\mathcal{L}(\mathcal{F}(E)) \otimes_{\tau_2} I_{H_2}$ . Further, the restriction of  $\Psi$  to the ultra-weak closure of  $\tau_1^{\mathcal{F}(E)}(H^\infty(E))$  is a completely isometric, ultra-weak homeomorphism from  $\tau_1^{\mathcal{F}(E)}(H^\infty(E))$  onto the ultra-weak closure of  $\tau_2^{\mathcal{F}(E)}(H^\infty(E))$ .*

**Proof.** For  $R \in \mathcal{L}(\mathcal{F}(E))$ ,  $\xi \in \mathcal{F}(E)$  and  $h \in H_i$ ,  $i = 1, 2$ ,  $\langle R\xi \otimes h, R\xi \otimes h \rangle = \langle h, \tau_i(\langle R\xi, R\xi \rangle)h \rangle$ . Thus  $R \otimes_{\tau_i} I_{H_i} = 0$  if and only if  $\langle R\xi, R\xi \rangle \subseteq \ker(\tau_i)$  for all  $\xi \in E$ . Since we assume that  $\ker(\tau_1) = \ker(\tau_2)$ , the map  $\Psi$  is a well defined, injective map. It is clearly a  $*$ -homomorphism and normality is also easy to check.  $\square$

We turn next to the problem of identifying the structure projection  $P$  for  $\mathcal{S}$ .

**Proposition 7.4** *If  $\Phi$  is the map defined by equation (27), then*

$$\mathcal{S} \cap \ker(\Phi) = \bigcap_{k=1}^{\infty} \mathcal{S}_0^k.$$

**Proof.** For  $x \in \mathcal{S}_0^k$  and  $\mathcal{M} \in \mathfrak{M}$ ,  $x^{(\infty)}[\mathcal{S}^{(\infty)}\mathcal{M}] \subseteq [(\mathcal{S}_0^k)^{(\infty)}\mathcal{M}]$ . Since  $\bigcap_k [(\mathcal{S}_0^k)^{(\infty)}\mathcal{M}] = \{0\}$ , we find that  $\bigcap_{k=1}^{\infty} \mathcal{S}_0^k \subseteq \ker(\Phi)$ . On the other hand, if  $x \in \mathcal{S}$  is not in  $\bigcap_{k=1}^{\infty} \mathcal{S}_0^k$ , then there is a  $k \geq 1$  such that  $x$  is not in  $\mathcal{S}_0^k$ . Consequently, there is an ultra-weakly continuous linear functional  $f$  such that  $f$  vanishes on  $\mathcal{S}_0^k$  but  $f(x) \neq 0$ . Since  $f$  is ultra-weakly continuous, we may find vectors  $\xi, \eta$  in  $H^{(\infty)}$  such that  $f(y) = \langle y^{(\infty)}\xi, \eta \rangle$  for all  $y \in B(H)$ . It follows that  $x^{(\infty)}\xi$  is not in  $[(\mathcal{S}_0^k)^{(\infty)}]$ , proving that  $\mathcal{N} := [\mathcal{S}^{(\infty)}\xi] \ominus [(\mathcal{S}_0^k)^{(\infty)}\xi] \neq \{0\}$ . In fact, we find that

$$x^{(\infty)}\mathcal{N} \not\subseteq [(\mathcal{S}_0^k)^{(\infty)}].$$

We can write  $\mathcal{N}$  as a direct sum of wandering spaces in  $\mathfrak{M}$ :

$$\begin{aligned} \mathcal{N} &= ([\mathcal{S}^{(\infty)}\xi] \ominus [(\mathcal{S}_0)^{(\infty)}\xi]) \oplus ([(\mathcal{S}_0)^{(\infty)}\xi] \ominus [(\mathcal{S}_0^2)^{(\infty)}\xi]) \oplus \dots \\ &\quad \oplus ([(\mathcal{S}_0^{k-1})^{(\infty)}\xi] \ominus [(\mathcal{S}_0^k)^{(\infty)}\xi]), \end{aligned}$$

which shows that  $x$  is not in the kernel of  $\Phi$ .  $\square$

**Proposition 7.5** *The space  $\bigcap_{k=1}^{\infty} \mathcal{S}_0^k$  is an ultra-weakly closed left ideal in the von Neumann algebra  $N$  generated by  $\mathcal{S}$ . Thus*

$$\bigcap_{k=1}^{\infty} \mathcal{S}_0^k = NP = \mathcal{S}P \tag{29}$$

for some projection  $P \in \mathcal{S} \cap \sigma(M)'$ .

Recall that  $P$  is called the structure projection for  $\mathcal{S}$  (and for  $(S, \sigma)$ .)

**Proof.** We first claim that the map  $\Phi$  defined in equation (27) satisfies the equation

$$\Phi(S(\xi)^*R) = (T_\xi^* \otimes I_\mathcal{E})\Phi(R), \quad \xi \in E, \quad R \in \mathcal{S}.$$

Since  $\Phi$  is an ultra-weakly continuous linear map, it suffices to prove the claim for  $R = \sigma(a)$ ,  $a \in M$ , and for  $R = S(\eta_1)S(\eta_2) \cdots S(\eta_k)$  for  $\eta_1, \eta_2, \dots, \eta_k \in E$ . For  $R = \sigma(a)$ ,  $\Phi(\sigma(a)) = \varphi_\infty(a) \otimes I_\mathcal{E}$  and, thus both sides of the equation are equal to 0. For  $R = S(\eta_1)S(\eta_2) \cdots S(\eta_k)$ , we have

$$\begin{aligned} \Phi(S(\xi)^*R) &= \Phi(\sigma(\langle \xi, \eta_1 \rangle)S(\eta_2) \cdots S(\eta_k)) \\ &= (\varphi_\infty(\langle \xi, \eta_1 \rangle) \otimes I_\mathcal{E})\Phi(S(\eta_2) \cdots S(\eta_k)) \\ &= (T_\xi^* \otimes I_\mathcal{E})(T_{\eta_1} \otimes I_\mathcal{E})\Phi(S(\eta_2) \cdots S(\eta_k)) \\ &= (T_\xi^* \otimes I_\mathcal{E})\Phi(R), \end{aligned}$$

which proves the claim. Consequently, for  $R \in \bigcap_{k=1}^\infty \mathcal{S}_0^k (= \ker \Phi \cap \mathcal{S})$ ,  $\Phi(S(\xi)^*R) = 0$  and, thus,  $S(\xi)^*R \in \bigcap_{k=1}^\infty \mathcal{S}_0^k$ . Clearly  $S(\xi)R \in \bigcap_{k=1}^\infty \mathcal{S}_0^k$  for such  $R$  and, therefore,  $\bigcap_{k=1}^\infty \mathcal{S}_0^k$  is an ideal in  $N$ . Since  $\ker(\Phi) \cap \mathcal{S}$  is an ideal in  $\mathcal{S}$ , it follows that  $P \in \sigma(M)'$ .  $\square$

**Lemma 7.6** *The structure projection  $P$  and the model projection  $e$  for the isometric representation  $(S, \sigma)$  are related:  $e$  is the smallest projection in  $M$  satisfying*

$$\sigma(e) = \bigvee \{v(I - P)v^* : v \text{ is a unitary in } \sigma(M)'\}.$$

*In particular,  $\sigma(e)$  is the central support in  $\sigma(M)$  of  $P^\perp$ . So, if  $P = 0$ , then  $e$  is the support projection of  $\sigma$ ; thus if  $P = 0$  and the restriction of  $S \times \sigma$  to  $M$  is faithful, then  $e = I$ .*

**Proof.** If  $z$  is a projection in  $M$  with  $\sigma(z) \in \mathcal{S}_0$  then, for  $k \geq 1$ ,  $\sigma(z) = \sigma(z)^k \in \mathcal{S}_0^k$  and, thus,  $\sigma(z) \in NP$  and  $\sigma(z) \leq P$ . The converse also holds since if  $\sigma(z) = \sigma(z)P$ , then  $\sigma(z) \in \mathcal{S}_0$ . Thus it follows from Lemma 7.2 that  $e^\perp$  is the largest projection  $z$  in  $M$  such  $\sigma(z) \leq P$ . Equivalently,  $e$  is the smallest projection  $p$  in  $M$  such that  $\sigma(p) \geq I - P$ . The statement of the lemma is now immediate.  $\square$

**Theorem 7.7** *Let  $(S, \sigma)$  be an isometric representation of  $(E, M)$ , write  $\mathcal{S}$  for the ultra-weak closure of  $(S \times \sigma)(\mathcal{T}_+(E))$  and let  $e$  be the model projection for  $\mathcal{S}$ . Suppose  $\tau$  is a normal representation of  $M$  on a Hilbert space  $K$  and that the support projection of  $\tau$  is  $e$ . Suppose, also, that there is given an ultra-weakly continuous, completely contractive homomorphism  $\psi$  from  $\mathcal{S}$  to  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  such that  $\psi \circ (S \times \sigma) = \tau^{\mathcal{F}(E)}$  on  $\mathcal{T}_+(E)$ . Then  $\psi$  is surjective and the map  $\tilde{\psi} : \mathcal{S}/\ker(\psi) \rightarrow \tau^{\mathcal{F}(E)}(H^\infty(E))$  obtained by passing to the quotient is a complete isometry and a homeomorphism with respect to the ultra-weak topologies on  $\mathcal{S}/\ker(\psi)$  and  $\tau^{\mathcal{F}(E)}(H^\infty(E))$ .*

**Proof.** Recall the unitary group  $\{W_t\}$  in  $\mathcal{L}(\mathcal{F}(E))$ , and the gauge automorphism group  $\{\gamma_t\}$ ,  $\gamma_t = \text{Ad}W_t$ , discussed in paragraph 2.9. Recall in particular that

$$\Sigma_k(a) := \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{|j| < k} \left(1 - \frac{|j|}{k}\right) e^{-int} \right) \gamma_t(a) dt$$

for  $a \in \mathcal{L}(\mathcal{F}(E))$ . We write  $W_t^{(n)}$  for  $\text{diag}(W_t, \dots, W_t)$ ,  $\gamma_t^{(n)}$  for  $\text{Ad}W_t^{(n)}$  and

$$\Sigma_k^{(n)}(A) := \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{|j| < k} \left(1 - \frac{|j|}{k}\right) e^{-ijt} \right) \gamma_t^{(n)}(A) dt,$$

for  $A \in M_n(\mathcal{L}(\mathcal{F}(E)))$ . We find that  $\Sigma_k^{(n)}$  is a contractive map and  $\Sigma_k^{(n)}(A) \rightarrow A$  in the ultra-weak topology as  $k \rightarrow \infty$ .

Now fix  $A \in M_n(\varphi_\infty(e)H^\infty(E)\varphi_\infty(e))$  and write  $X_k := (S \times \sigma)^{(n)}(\Sigma_k^{(n)}(A))$ . Note that  $\Sigma_k^{(n)}(A) \in M_n(\mathcal{T}_+(E))$  so that the last expression is well defined and  $\psi^{(n)}(X_k) = (\tau^{\mathcal{F}(E)})^{(n)}(\Sigma_k^{(n)}(A))$ . Thus,  $\|X_k\| = \|(S \times \sigma)^{(n)}(\Sigma_k^{(n)}(A))\| \leq \|\Sigma_k^{(n)}(A)\| \leq \|A\|$  (here we used the fact that  $(S \times \sigma)^{(n)}$  is contractive on  $\varphi_\infty(e)\mathcal{T}_+(E)\varphi_\infty(e)$  and  $\Sigma_k^{(n)}(A)$  lies in  $M_n(\varphi_\infty(e)\mathcal{T}_+(E)\varphi_\infty(e))$ ). Now, the sequence  $X_k$  has a subnet  $X_{k_\alpha}$  that is ultra-weakly convergent to some  $X \in M_n(\mathcal{S})$  with  $\|X\| \leq \|A\|$ . Since  $\psi$  is ultra-weakly continuous, it follows that

$$\psi^{(n)}(X) = \lim \psi^{(n)}(X_{k_\alpha}) = \lim (\tau^{\mathcal{F}(E)})^{(n)}(\Sigma_{k_\alpha}^{(n)}(A)) = (\tau^{\mathcal{F}(E)})^{(n)}(A),$$

where the limits are all taken in the ultra-weak topology. Thus  $\psi^{(n)}$  is surjective for every  $n \geq 1$ .

Since  $e$  is  $E$ -saturated, Theorem 6.9 implies that  $\tau^{\mathcal{F}(E)}$ , restricted to  $H^\infty(E)\varphi_\infty(e)$ , is a complete isometry and, thus,

$$\|A\| \geq \|X\| \geq \|\psi^{(n)}(X)\| = \|(\tau^{\mathcal{F}(E)})^{(n)}(A)\|.$$

Therefore, in this case we get

$$\|X\| = \|(\tau^{\mathcal{F}(E)})^{(n)}(A)\|.$$

This shows that the map  $\tilde{\psi}$  from  $\mathcal{S}/\ker(\psi)$  onto  $\tau^{\mathcal{F}(E)}(H^\infty(E))$ , induced by  $\psi$ , is a complete isometry.

The fact that this map is an ultra-weak homeomorphism follows from the ultra-weak continuity of  $\psi$  as in the proof of [2, Theorem 1.1]. To be more precise, the map  $\tilde{\psi}$  is the dual of a map  $\psi_*$  from the predual of  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  to the predual of  $\mathcal{S}/\ker(\psi)$ . This map is injective (as  $\tilde{\psi}$  is surjective) and contractive and, thus, has a bounded inverse  $(\psi_*)^{-1}$ . The dual of this inverse is the inverse of  $\tilde{\psi}$ . Thus  $\tilde{\psi}$  is an ultra-weak homeomorphism.  $\square$

The following theorem is our analogue of [2, Theorem 2.6].

**Theorem 7.8** *Let  $(S, \sigma)$  be an isometric representation of  $(E, M)$  on  $H$ , let  $\mathcal{S}$  be the ultra-weak closure of  $(S \times \sigma)(\mathcal{T}_+(E))$ , let  $N$  be the von Neumann algebra generated by  $\mathcal{S}$ , and let  $P$  be the structure projection for  $\mathcal{S}$ . Then in addition to equation (29), the following assertions hold:*

- (1)  $P^\perp H$  is invariant for  $\mathcal{S}$ .
- (2)  $\mathcal{S}P^\perp = P^\perp \mathcal{S}P^\perp$ .
- (3)  $\mathcal{S} = NP + P^\perp \mathcal{S}P^\perp$ .
- (4) *If  $\tau$  is any normal representation of  $M$  whose support projection is the model projection for  $\mathcal{S}$ , then the map  $\Psi$  from  $(S \times \sigma)(\mathcal{T}_+(E))P^\perp$  to  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  defined by the equation*

$$\Psi((S \times \sigma)(a)P^\perp) = \tau^{\mathcal{F}(E)}(a), \quad a \in \mathcal{T}_+(E),$$

*extends to a complete isometric isomorphism from  $\mathcal{S}P^\perp$  onto  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  that is a homeomorphism with respect to the ultra-weak topologies on  $\mathcal{S}P^\perp$  and  $\tau^{\mathcal{F}(E)}(H^\infty(E))$ .*

**Proof.** Suppose first that  $\tau$  is the representation defined in (26) and that  $\Phi$  is the map defined in (28). Since  $\ker(\Phi)$  is an ideal in  $\mathcal{S}$  and the structure projection,  $P$ , belongs to it,  $P\mathcal{S}P^\perp \subseteq \ker(\Phi) = NP$ . But then  $P\mathcal{S}P^\perp = 0$ . Thus  $P^\perp H$  is invariant for  $\mathcal{S}$  and  $\mathcal{S}P^\perp = P^\perp \mathcal{S}P^\perp$ .

Now,  $\Phi$  is a completely contractive, ultra-weakly continuous, homomorphism mapping  $\mathcal{S}$  into the  $\sigma$ -weak closure of  $\tau^{\mathcal{F}(E)}(\mathcal{T}_+(E))$ . So we may apply Theorem 7.7 to  $\Phi$  (which plays the role of  $\psi$  in the statement of Theorem 7.7) to conclude that the map induced by  $\Phi$  on  $\mathcal{S}P^\perp \simeq \mathcal{S}/\ker(\Phi)$  is completely isometric and an ultra-weak homeomorphism onto the ultra-weak closure of  $\tau^{\mathcal{F}(E)}(\mathcal{T}_+(E))$  - which is  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  by Theorem 6.9 and the fact that the model projection for  $\mathcal{S}$  is  $E$ -saturated, Lemma 7.2. But once we have proven the result with the special representation  $\tau$  from (26), we can replace it with any representation with the same support, viz. the model projection of  $\mathcal{S}$ , thanks to Proposition 7.3.  $\square$

To connect the structure projection with the absolutely continuous subspace of an isometric representation, we employ the universal isometric representation  $(S_0, \sigma_0)$  from paragraph 2.11 and use results developed in Section 3. Recall that  $(S_0, \sigma_0)$  is the representation induced by a representation  $\pi$  on a Hilbert space  $K_0$ , and so  $(S_0, \sigma_0)$  acts on the Hilbert space  $\mathcal{F}(E) \otimes_\pi K_0$ .

**Theorem 7.9** *Let  $(S, \sigma)$  be an isometric representation of  $(E, M)$  on a Hilbert space  $H$ , let  $P$  be its structure projection, and let  $Q$  be the structure projection for the representation  $(S \times \sigma) \oplus (S_0 \times \sigma_0)$ , acting on  $H \oplus \mathcal{F}(E) \otimes_\pi K_0$ . Then*

- (1)  $\mathcal{V}_{ac}(S, \sigma) = H \ominus Q(H \oplus (\mathcal{F}(E) \otimes_\pi K_0))$ ,
- (2)  $P^\perp H \subseteq \mathcal{V}_{ac}(S, \sigma)$ , and
- (3) *the range of  $Q$  is contained in the range of  $P$ , viewed as a subspace of  $H \oplus (\mathcal{F}(E) \otimes_\pi K_0)$ .*

**Proof.** We shall write  $\mathcal{S}$  for the ultra-weak closure of  $S \times \sigma(\mathcal{T}_+(E))$ . To simplify the notation, we shall write  $\rho$  for  $(S \times \sigma) \oplus (S_0 \times \sigma_0)$  and we shall write  $\mathcal{S}_\rho$  for the ultra-weak closure of  $\rho(\mathcal{T}_+(E))$ . We first want to prove that the range of  $Q$  is contained in  $H$ . For this purpose and for later use, we shall write  $P_H$  for the projection of  $H \oplus (\mathcal{F}(E) \otimes_\pi K_0)$  onto  $H$ . Since  $\rho(a)$  commutes with  $P_H$ , for every  $a \in \mathcal{T}_+(E)$ , the von Neumann algebra generated by  $\mathcal{S}_\rho$ ,  $N_\rho$ , commutes with  $P_H$ . In particular,  $Q$  commutes with  $P_H$ . So it

suffices to show that if  $\xi$  is a vector in  $H \oplus (\mathcal{F}(E) \otimes_\pi K_0)$  such that  $\xi = Q\xi$  lies in  $H^\perp$ , then  $\xi = 0$ . So fix such a vector  $\xi$ . Then  $\xi \in \mathcal{F}(E) \otimes_\pi K_0$  and, for every  $\eta \in E^{\otimes m}$ ,  $\rho(T_\eta)\xi \in (E^{\otimes m} \oplus E^{\otimes(m+1)} \oplus \dots) \otimes_\pi K_0$ . Thus, for every  $X \in (\mathcal{S}_\rho)_0^m$ ,  $X\xi \in (E^{\otimes m} \oplus E^{\otimes(m+1)} \oplus \dots) \otimes_\pi K_0$ . It follows that, if  $X \in \bigcap_m (\mathcal{S}_\rho)_0^m = N_\rho Q$ , then  $X\xi = 0$ . In particular,  $Q\xi = 0$ . This shows that the range of  $Q$  is contained in  $H$ .

Next observe that the wandering vectors of *any* isometric representation are orthogonal to the range of its structure projection. To prove this, we will work with  $(S, \sigma)$ , but the argument is quite general. So suppose  $\xi \in H$  is a wandering vector for  $(S, \sigma)$ . Then  $\mathcal{M}_0 := [\sigma(M)\xi]$  is a  $\sigma(M)$ -invariant wandering subspace for  $(S, \sigma)$ ,  $\mathcal{M} := \{\eta^{(\infty)} : \eta \in \mathcal{M}_0\}$  is in  $\mathfrak{M}$ , and  $\xi^{(\infty)} \in [\mathcal{S}^{(\infty)}\mathcal{M}]$ . Since the structure projection  $P$  of  $(S, \sigma)$  lies in the kernel of  $\Phi$  by Proposition 7.4,  $P^{(\infty)}|[\mathcal{S}^{(\infty)}\mathcal{M}] = 0$  and, thus,  $P\xi = 0$ . Thus the span of the wandering vectors for  $\rho$  is orthogonal to the range of  $Q$ . But by Corollary 3.8, the span of the wandering vectors for  $\rho$  is  $\mathcal{V}_{ac}(S, \sigma) \cap H$ . Thus  $\mathcal{V}_{ac}(S, \sigma) \subseteq P_H Q^\perp(H \oplus (\mathcal{F}(E) \otimes K))$ .

To show the reverse inclusion, suppose  $\xi = P_H Q^\perp \xi$ . Then  $\xi = Q^\perp \xi$ , as  $Q \leq P_H$ . Let  $f$  be the linear functional on  $\mathcal{T}_+(E)$  defined by the equation  $f(a) = \langle \rho(a)\xi, \xi \rangle = \langle \rho(a)Q^\perp \xi, Q^\perp \xi \rangle$ ,  $a \in \mathcal{T}_+(E)$ . We want to use Theorem 7.8 to show that  $\xi$  lies in  $\mathcal{V}_{ac}(S, \sigma)$  by showing that  $f$  extends to an ultra-weakly continuous linear functional on  $H^\infty(E)$ . In the notation of that theorem, replace  $S \times \sigma$  with  $\rho$  and let  $e_\rho$  be the model projection for  $\rho$ . Also, let  $\tau$  be any normal representation of  $M$  with support equal  $e_\rho$ . Then by part (4) of Theorem 7.8 there is a completely isometric isomorphism  $\Psi$  from  $\mathcal{S}_\rho Q^\perp$  onto  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  that is an ultra-weak homeomorphism such that  $\Psi(\rho(a)Q^\perp) = \tau^{\mathcal{F}(E)}(a)$  for all  $a \in H^\infty(E)$ . We conclude, then, that  $f(a) = \langle \rho(a)Q^\perp \xi, Q^\perp \xi \rangle = \langle \Psi^{-1} \circ \tau^{\mathcal{F}(E)}(a)\xi, \xi \rangle$ ,  $a \in \mathcal{T}_+(E)$ , extends to an ultra-weakly continuous linear functional on  $H^\infty(E)$ . By Remark 3.2,  $\xi \in \mathcal{V}_{ac}(S, \sigma)$ . Thus we conclude that  $\mathcal{V}_{ac}(S, \sigma) = H \ominus Q(H \oplus (\mathcal{F}(E) \otimes_\pi K_0))$ . This proves point (1).

The argument needed to prove that  $P^\perp H \subseteq \mathcal{V}_{ac}(S, \sigma)$  is similar. Let  $\xi \in P^\perp H$  and consider the functional  $g(a) := \langle (S \times \sigma)(a)\xi, \xi \rangle = \langle (S \times \sigma)(a)P^\perp \xi, P^\perp \xi \rangle$  for  $a \in \mathcal{T}_+(E)$ . Using part (4) of Theorem 7.8 (and the notation there), we may write  $g(a) = \langle \Psi^{-1}(\tau_e^{\mathcal{F}(E)}(a))\xi, \xi \rangle$  for any normal representation  $\tau_e$  of  $M$  whose support projection is the model projection for  $S \times \sigma$ . This shows that  $g$  is ultra-weakly continuous and so  $\xi \in \mathcal{V}_{ac}(S, \sigma)$ . Thus  $P^\perp H$  is contained in  $\mathcal{V}_{ac}(S, \sigma)$ , proving point (2).



Since  $P^\perp H \subseteq \mathcal{V}_{ac}(S, \sigma) = H \ominus Q(H \oplus (\mathcal{F}(E) \otimes_\pi K_0))$ , we see that the range of  $P_H Q$  is contained in the range of  $P$ . Since, however, the range of  $Q$  is contained in  $H$ , as we proved at the outset, we conclude that the range of  $Q$  is contained in the range of  $P$ , proving point (3).  $\square$

As a corollary we conclude with following theorem, which complements Theorem 3.10 in the sense that if  $(S, \sigma)$  is an ultra-weakly continuous, isometric representation of  $(E, M)$ , then  $(S, \sigma)$  is absolutely continuous if and only if  $S \times \sigma \oplus S_0 \times \sigma_0$  is “completely isometrically equivalent” to an induced representation of  $(E, M)$ . Thus,  $(S, \sigma)$  is absolutely continuous if and only if it “equivalent to a subrepresentation of an induced representation”.

**Theorem 7.10** *Let  $(S, \sigma)$  be an ultra-weakly continuous, isometric covariant representation of  $(E, M)$  on a Hilbert space and denote by  $\rho$  the representation  $S \times \sigma \oplus S_0 \times \sigma_0$ . Then the following assertions are equivalent:*

- (1)  *$(S, \sigma)$  is absolutely continuous.*
- (2) *The structure projection for  $\rho$  is zero.*
- (3) *If  $\tau$  is any representation of  $M$  whose support projection is the model projection of  $S \times \sigma$ , then there is an ultra-weakly homeomorphic, completely isometric isomorphism  $\Psi$  from  $\tau^{\mathcal{F}(E)}(H^\infty(E))$  onto the ultra-weak closure  $\mathcal{S}_\rho$  of  $\rho(\mathcal{T}_+(E))$  such that  $\Psi \circ \tau^{\mathcal{F}(E)}(a) = \rho(a)$ , for all  $a \in \mathcal{T}_+(E)$ .*

**Proof.** Let  $P$  be the structure projection for  $S \times \sigma$ . If  $(S, \sigma)$  is absolutely continuous then  $\mathcal{V}_{ac}(S, \sigma) = H$  and it follows from Theorem 7.9 (1) that the range of  $Q$  is orthogonal to  $H$ . Since, however, the range of  $Q$  is contained in the range of  $P$  by part (3) of that lemma, we conclude that  $Q = 0$ . This proves that (1) implies (2). The implication, (2)  $\Rightarrow$  (3), follows from Theorem 7.8 (4). Now assume that (3) holds. Then  $\Psi^{-1}(\mathcal{S}_\rho) = \tau^{\mathcal{F}(E)}(H^\infty(E))$ , and for  $m \geq 0$ ,  $\Psi^{-1}((\mathcal{S}_\rho)_0^m)$  is the image under  $\tau^{\mathcal{F}(E)}$  of the space  $H_m^\infty(E)$ , which consists of all operators  $X \in H^\infty(E)$  with the property that  $\Phi_k(X) = 0$ ,  $k \leq m$ , where  $\Phi_k$  is the  $k^{th}$  Fourier operator (8). Then  $\Psi^{-1}(\bigcap_m (\mathcal{S}_\rho)_0^m) = \bigcap_m \Psi^{-1}((\mathcal{S}_\rho)_0^m) = \bigcap_m \tau^{\mathcal{F}(E)}(H_m^\infty(E)) = \tau^{\mathcal{F}(E)}(\bigcap_m H_m^\infty(E))$ . Since  $\bigcap_m H_m^\infty(E) = \{0\}$ ,  $\bigcap_m (\mathcal{S}_\rho)_0^m = \{0\}$  and so  $Q = 0$ . This proves that (3) implies (2). Since Theorem 7.9 (1) shows that (2) implies (1), the proof is complete.  $\square$

**Remark 7.11** In [12], M. Kennedy deftly uses technology from the theory of dual operator algebras to show that in the setting when  $M = \mathbb{C}$  and  $E = \mathbb{C}^d$ , the ultra-weak closure of the image of an isometric representation of  $\mathcal{T}_+(\mathbb{C}^d)$  is ultraweakly homeomorphic and completely isometrically isomorphic to  $H^\infty(\mathbb{C}^d)$  if and only if the representation is absolutely continuous. Whether his result can be generalized to the setting of this paper remains to be seen.

## References

- [1] W. Arveson, *Subalgebras of  $C^*$ -algebras*, Acta Math. **123** (1969), 141–224.
- [2] K. Davidson, E. Katsoulis and D. Pitts, *The structure of free semigroup algebras*, J. Reine Angew. Math. **533** (2001), 99–125.
- [3] K. Davidson, J. Li and D. Pitts, *Absolutely continuous representations and a Kaplansky density theorem for free semigroup algebras*, J. Functional Anal. **224** (2005), 160 – 191.
- [4] J. Dixmier, *Von Neumann Algebras*, North-Holland Mathematical Library vol. **27**, North Holland Amsterdam New York oxford, 1981.
- [5] R. G. Douglas, *On the operator equation  $S^*XT = X$  and related topics*, Acta Sci. Math. (Szeged) **30** (1969), 19–32.
- [6] R. G. Douglas, *On majorization, factorization, and range inclusion of operators in Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415
- [7] N. Fowler, P.S. Muhly and I. Raeburn, *Representations of Cuntz-Pimsner algebras*, Ind. Univ. Math. J. **52** (2003), 569 - 605.
- [8] N. Fowler and I. Raeburn, *The Toeplitz algebra of a Hilbert bimodule*, Ind. Univ. Math. J. **48** (1999), 155-181.
- [9] F. Gantmacher, *Applications of the Theory of Matrices*, Interscience Publishers, Wiley, New York, 1959.
- [10] R.V. Kadison and J.R. Ringrose, *Fundamentals of the theory of operator algebras* vol. 1, Academic Press 1983.

- [11] T. Kato, *Smooth operators and commutators*, Studia Math. **31** (1968), 535–546.
- [12] M. Kennedy, *Absolutely continuous presentations of the non-commutative disk algebra*, preprint (arXiv: 1001.3182v1).
- [13] V. Manuilov and E. Troitsky, *Hilbert  $C^*$ -modules*, Translations of Mathematical Monographs, Vol. **226**, Amer. Math. Soc., Providence, 2005.
- [14] P. S. Muhly and B. Solel, *The Poisson kernel for Hardy algebras*, Complex Analysis and Operator Theory **3** (2009), 221–242.
- [15] P. S. Muhly and B. Solel, *Hardy algebras,  $W^*$ -correspondences and interpolation theory*, Math. Ann. **330** (2004), 353–415.
- [16] P. S. Muhly and B. Solel, *Quantum Markov processes (correspondences and dilations)*, Int. J. Math. **13** (2002), 863–906.
- [17] P. S. Muhly and B. Solel, *Tensor algebras, induced representations, and the Wold decomposition*, Canad. J. Math. **51** (1999), 850–880.
- [18] P. S. Muhly and B. Solel, *Tensor algebras over  $C^*$ -correspondences (Representations, dilations, and  $C^*$ -envelopes)*, J. Functional Anal. **158** (1998), 389–457.
- [19] V. Paulsen, *Every completely polynomially bounded operator is similar to a contraction*, J. Funct. Anal. **55** (1984), 1–17.
- [20] G. Popescu, *Similarity and ergodic theory of positive linear maps*, J. reine angew. Math. **561** (2003), 87–129.
- [21] M. Rieffel, *Morita equivalence for  $C^*$ -algebras and  $W^*$ -algebras*, J. Pure Appl. Alg. **5** (1974), 51–96.
- [22] M. Rieffel, *Induced representations of  $C^*$ -algebras*, Adv. in Math. **13** (1974), 176–257.
- [23] D. Revuz, *Markov Chains*, North Holland, Elsevier, New York 1984.
- [24] O. Shalit and B. Solel, *Subproduct Systems*, Doc. Math. **14** (2009), 801–868.

- [25] F. Stinespring, *Positive functions on  $C^*$ -algebras*, Proc. Amer. Math. Soc. 6 (1955), 211–216.
- [26] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*,
- [27] M. Takesaki, *Theory of Operator Algebras I*, Springer-Verlag, New York, Heidelberg, and Berlin, 1979.